Chapter 9

Fresnel Reflection

9.0.1 \( \pi \) polarization:

We have from the law of reflection that

\[ \theta_i = \theta_r \quad (9.1) \]

We have from Snell’s law that

\[ n_1 \sin \theta_i = n_2 \sin \theta_t \quad (9.2) \]
Boundary conditions for the tangential components of $\mathbf{H}$ give us

$$H_i + H_r = H_t$$  \hspace{1cm} (9.3)

or

$$\frac{E_i}{Z_1} + \frac{E_r}{Z_1} = \frac{E_t}{Z_2}$$  \hspace{1cm} (9.4)

or

$$E_t = \frac{Z_2}{Z_1} (E_i + E_r)$$

and

$$E_i \cos \theta_i - E_r \cos \theta_r = E_t \cos \theta_t$$  \hspace{1cm} (9.5)

and

$$E_i \cos \theta_i - E_r \cos \theta_i = \frac{Z_2}{Z_1} (E_i + E_r) \cos \theta_t$$  \hspace{1cm} (9.6)

and defining

$$r = \frac{E_r}{E_i}$$  \hspace{1cm} (9.7)

we have

$$\cos \theta_i - r \cos \theta_r = \frac{Z_2}{Z_1} (1 + r_r) \cos \theta_t$$  \hspace{1cm} (9.8)

and

$$\cos \theta_i - \frac{Z_2}{Z_1} \cos \theta_t = r (\cos \theta_i + \frac{Z_2}{Z_1} \cos \theta_t)$$  \hspace{1cm} (9.9)

and finally

$$r = \frac{Z_1 \cos \theta_i - Z_2 \cos \theta_t}{Z_1 \cos \theta_t + Z_2 \cos \theta_t}$$  \hspace{1cm} (9.10)

If the material is non-magnetic, then

$$r = \frac{n_2 \cos \theta_i - n_1 \cos \theta_t}{n_2 \cos \theta_i + n_1 \cos \theta_t}$$  \hspace{1cm} (9.11)

and writing $\theta_t$ explicitly, we get

$$r = \frac{n_2 \cos \theta_i - n_1 \cos \theta_t}{n_2 \cos \theta_i + n_1 \cos \theta_t} \sqrt{1 - \frac{n_1^2}{n_2^2} \sin^2 \theta_i}$$  \hspace{1cm} (9.12)

Plotting this for glass, with $n_1 = 1$, $n_2 = 1.5$, gives
We note that $r$ goes to zero at the Brewster angle; here $\tan \theta_i = \frac{n_2}{n_1}$.

We also define the transmission coefficient

$$t = \frac{E_t}{E_i}$$ \hspace{1cm} (9.13)

which, since

$$E_t = \frac{Z_2}{Z_1} (E_i + E_r)$$ \hspace{1cm} (9.14)

becomes

$$t = \frac{Z_2}{Z_1} (1+r) = \frac{Z_2}{Z_1} \left(1 + \frac{Z_1 \cos \theta_i - Z_2 \cos \theta_i}{Z_1 \cos \theta_i + Z_2 \cos \theta_i}\right) = \frac{2Z_2 \cos \theta_i}{Z_1 \cos \theta_i + Z_2 \cos \theta_i} = \frac{2n_1 \cos \theta_i}{n_2 \cos \theta_i + n_1 \cos \theta_i}$$ \hspace{1cm} (9.15)

### 9.0.2 $\sigma$ Polarization

For $\sigma$ polarization, we have
and here

\[ E_i + E_r = E_t \quad (9.16) \]

or

\[ \frac{H_i}{Z_1^{-1}} + \frac{H_r}{Z_1^{-1}} = \frac{H_t}{Z_2^{-1}} \quad (9.17) \]

or

\[ H_t = \frac{Z_2^{-1}}{Z_1^{-1}}(H_i + H_r) \]

and

\[ H_i \cos \theta_i - H_r \cos \theta_r = H_t \cos \theta_t \quad (9.18) \]

and

\[ H_i \cos \theta_i - H_r \cos \theta_i = \frac{Z_2^{-1}}{Z_1^{-1}}(H_i + H_r) \cos \theta_t \quad (9.19) \]

and defining

\[ r = \frac{E_r}{E_i} = \frac{H_r}{H_i} \quad (9.20) \]

, we have

\[ \cos \theta_i - r \cos \theta_i = \frac{Z_2^{-1}}{Z_1^{-1}}(1 + r_r) \cos \theta_t \quad (9.21) \]

and

\[ \cos \theta_i - \frac{Z_2^{-1}}{Z_1^{-1}} \cos \theta_t = r(\cos \theta_i + \frac{Z_2^{-1}}{Z_1^{-1}} \cos \theta_t) \quad (9.22) \]
and finally

\[ r = \frac{Z_2 \cos \theta_i - Z_1 \cos \theta_t}{Z_2 \cos \theta_i + Z_1 \cos \theta_t} \]  

(9.23)

If the material is non-magnetic, then

\[ r = \frac{n_1 \cos \theta_i - n_2 \cos \theta_t}{n_1 \cos \theta_i + n_2 \cos \theta_t} \]  

(9.24)

and writing \( \theta_t \) explicitly, we get

\[ r = \frac{n_1 \cos \theta_i - n_2 \cos \theta_t}{n_1 \cos \theta_i + n_2 \cos \theta_t} = \frac{\sqrt{1 - \frac{n_2^2}{n_1^2} \sin^2 \theta_i}}{n_1 \cos \theta_i + n_2 \sqrt{1 - \frac{n_2^2}{n_1^2} \sin^2 \theta_i}} \]  

(9.25)

The transmittance is

\[ t = \frac{E_t}{E_i} \]  

(9.26)

and since

\[ E_i + E_r = E_t \]  

(9.27)

we have

\[ t = 1 + r = 1 + \frac{Z_2 \cos \theta_i - Z_1 \cos \theta_t}{Z_2 \cos \theta_i + Z_1 \cos \theta_t} = \frac{2Z_2 \cos \theta_i}{Z_2 \cos \theta_i + Z_1 \cos \theta_t} \]

\[ = \frac{2n_1 \cos \theta_i}{n_1 \cos \theta_i + n_2 \cos \theta_t} \]

Summary
At an interface between two materials, we have, for $\pi$ polarization

\[ r_{12}^\pi = \frac{n_2 \cos \theta_1 - n_1 \cos \theta_2}{n_2 \cos \theta_1 + n_1 \cos \theta_2} \]  \hspace{1cm} (9.28)

and

\[ t_{12}^\pi = \frac{2n_1 \cos \theta_1}{n_2 \cos \theta_1 + n_1 \cos \theta_2} \]  \hspace{1cm} (9.29)

while for $\sigma$ polarization, we have

\[ r_{12}^\sigma = \frac{n_1 \cos \theta_1 - n_2 \cos \theta_2}{n_1 \cos \theta_1 + n_2 \cos \theta_2} \]  \hspace{1cm} (9.30)

and

\[ t_{12}^\sigma = \frac{2n_1 \cos \theta_1}{n_1 \cos \theta_1 + n_2 \cos \theta_2} \]  \hspace{1cm} (9.31)

### 9.0.3 Total Internal Reflection (TIR)

**$\sigma$ polarization**

Consider the reflection coefficient for $\sigma$ polarization.

\[ r_{12}^\sigma = \frac{n_1 \cos \theta_1 - n_2 \cos \theta_2}{n_1 \cos \theta_1 + n_2 \cos \theta_2} \]  \hspace{1cm} (9.32)

Using Snell’s Law, we can write

\[ \sin \theta_2 = \frac{n_1}{n_2} \sin \theta_1 \]

\[ \cos \theta_2 = \sqrt{1 - \frac{n_1^2}{n_2^2} \sin^2 \theta_1} \]  \hspace{1cm} (9.33)

\[ r_{12}^\sigma = \frac{n_1 \cos \theta_1 - n_2 \sqrt{1 - \frac{n_1^2}{n_2^2} \sin^2 \theta_1}}{n_1 \cos \theta_1 + n_2 \sqrt{1 - \frac{n_1^2}{n_2^2} \sin^2 \theta_1}} \]  \hspace{1cm} (9.34)

Now suppose that $n_1 > n_2$. In this case, if

\[ \sin \theta_1 > \frac{n_2}{n_1} \]  \hspace{1cm} (9.35)
we have

\[ \cos \theta_2 = i \sqrt{\frac{n_1^2}{n_2^2} \sin^2 \theta_1 - 1} \]  

(9.36)

and \( \cos \theta_2 \) is imaginary (but \( \sin \theta_t \) remains real), and \( r_{12}^\sigma \) is complex. It is useful to define the critical angle for TIR as

\[ \sin \theta_c = \frac{n_2}{n_2} \]  

(9.37)

or

\[ \theta_c = \sin^{-1} \frac{n_2}{n_1} \]  

(9.38)

Now we can write

\[ r_{12}^\sigma = \frac{n_1 \cos \theta_1 - n_2 i \sqrt{\frac{n_1^2}{n_2^2} \sin^2 \theta_1 - 1}}{n_1 \cos \theta_1 + n_2 i \sqrt{\frac{n_1^2}{n_2^2} \sin^2 \theta_1 - 1}} = \frac{a - ib}{a + ib} \]  

(9.39)

or

\[ r_{12}^\sigma = |r_{12}^\sigma| e^{i\phi} \]  

(9.40)

We write

\[ r_{12}^\sigma = \frac{a - ib}{a + ib} \]  

(9.41)

and note that

\[ |r_{12}^\sigma| = \sqrt{r_{12}^\sigma r_{12}^{\sigma*}} = 1 \]  

(9.42)

So the magnitude of the reflection coefficient is 1. Writing

\[ r_{12}^\sigma = \frac{a - ib}{a + ib} = \frac{(a - ib)^2}{a^2 + b^2} = \frac{a^2 - b^2}{a^2 + b^2} - \frac{2iab}{a^2 + b^2} \]  

(9.43)

the phase \( \phi \) can be conveniently written as

\[ \tan \phi = -\frac{2ab}{a^2 - b^2} = -\frac{2n_1 \cos \theta_1 n_2 \sqrt{\frac{n_1^2}{n_2^2} \sin^2 \theta_1 - 1}}{n_1 \cos^2 \theta_1 - n_2^2 (\frac{n_1^2}{n_2^2} \sin^2 \theta_1 - 1)} = -\frac{2n_1 \cos \theta_1 \sqrt{\frac{n_1^2}{n_2^2} \sin^2 \theta_1 - 1}}{n_1^2 (\cos^2 \theta_1 - \sin^2 \theta_1) + n_2^2} \]

The reflected wave has the same amplitude as the incident wave, but there is an angle-dependent phase shift. The reflection coefficient for the intensity is

\[ R^\sigma = r_{12}^\sigma r_{12}^{\sigma*} = 1 \]  

(9.44)
All of the incident power is reflected - this is total internal reflection. No power is transmitted.

The transmission coefficient is

$$t_{12}^2 = \frac{2n_1 \cos \theta_1}{n_1 \cos \theta_1 + n_2 \cos \theta_2} = \frac{2n_1 \cos \theta_1}{n_1 \cos \theta_1 + n_2 i \sqrt{\frac{n_2^2}{n_1^2} \sin^2 \theta_1 - 1}}$$

(9.45)

a complex number, which equals 2 at the critical angle of incidence $\theta_c$.

It is interesting to look at the wavevector $k_2$. From Maxwell’s equations, we have (assuming isotropy and nonmagnetic materials) that

$$k_2^2 = \omega^2 \mu \varepsilon$$

(9.46)

or

$$k_2 = \frac{\omega n_2}{c}$$

(9.47)

The in-plane component of $k_2$ (in the interface)

$$k_{2x} = k_2 \sin \theta_2 = k_1 \sin \theta_1$$

(9.48)

and, since $\sin \theta_2$ is real, the in-plane component of the wave-vector $k_2$ is real. Furthermore, $\theta_2$ is such that Snell’s law is satisfied;

$$k_2 \sin \theta_2 = k_1 \sin \theta_1$$

(9.49)

We also have for the normal component $k_{2z}$ of $k_2$ that

$$k_{2z} = k_2 \cos \theta_2$$

(9.50)

but this is imaginary, since

$$k_{2z} = k_2 i \sqrt{\frac{n_1^2}{n_2^2} \sin^2 \theta_1 - 1}$$

(9.51)

so we have an imaginary normal component of the wave vector. We can also write this as

$$k_{2z} = i \frac{n_1}{n_2} k_2 \sqrt{\sin^2 \theta_1 - \sin^2 \theta_c} = i k_1 \sqrt{\sin^2 \theta_1 - \sin^2 \theta_c}$$

In essence, what happens is the following. As the angle of incidence $\theta_1$ increases, the angle of transmission $\theta_2$ grows. At the critical angle,
\[ \theta_1 = \theta_c, \text{ } k_2 \text{ is in the plane of the interface. As the angle of incidence increases further, an imaginary component of } k_2 \text{ appears, normal to the interface. This increases the length of } k_2 \text{ sufficiently so that its projection on the interface can match that of the incident wave.} \]

Note that
\[ k_2^2 = k_2 \cdot k_2 = k_{2x}^2 - |k_{2z}|^2 \]  
(9.52)

So, indeed, having an imaginary component allows \( k_2 \) to be longer. Substitution gives
\[ k_1^2 \sin^2 \theta_1 - k_2^2 \left( \frac{n_2^2}{n_1^2} \sin^2 \theta_1 - 1 \right) = k_2^2 = \frac{\omega n_2}{c} \]
(9.53)
as expected.

The transmitted electric field at the interface is
\[ E_t = E_i t_{12} \]
(9.54)
so it is just the incident field multiplied by a complex amplitude. Explicitly, we have
\[ E_t = E_{ot} \hat{y} e^{i(k_2 \cdot r - \omega t)} = E_{ot} \hat{y} e^{-k_{2z} z} e^{i(k_2 x - \omega t)} \]
(9.55)
so we have a real wave propagating along the interface, with wave vector
\[ k_{2x} = k_2 \sin \theta_2 = k_1 \sin \theta_1 \]
(9.56)
and an exponentially decaying envelope normal to the interface with decay length
\[ \zeta = \frac{1}{k_{2z}} = \frac{1}{k_1 \sqrt{\sin^2 \theta_1 - \sin^2 \theta_c}} = \frac{\lambda_o}{2 \pi n_1 \sqrt{\sin^2 \theta_1 - \sin^2 \theta_c}} \]  
(9.57)

The \( \mathbf{H} \) field is given, as usual, by
\[ \mathbf{H}_t = \frac{1}{\omega \mu_o} k_2 \times E_t \]
(9.58)
or
\[ \mathbf{H}_t = \frac{1}{Z_2} \hat{k}_2 \times E_t \]
(9.59)
and
\[ \mathbf{H}_t = \frac{1}{Z_2} (\sin \theta_2 \hat{x} + \cos \theta_2 \hat{z}) \times E_t \hat{y} = \frac{1}{Z_2} E_t \sin \theta_2 \hat{z} - \frac{1}{Z_2} E_t \cos \theta_2 \hat{x} \]
(9.60)
so $\mathbf{H}_t$ has a real component along the $z-$ direction, and an imaginary component along the $x$ direction.

Is this still a transverse wave?

Is the relation

$$\nabla \cdot \mathbf{H} = k_2 \cdot \mathbf{H}_t = 0$$

satisfied? Substitution gives

$$k_2 \cdot \mathbf{H}_t = (\frac{1}{Z_2} E_t \sin \theta_2 \hat{z} - \frac{1}{Z_2} E_t \cos \theta_2 \hat{x}) \cdot (k_2 \cos \theta_2 \hat{z} + k_2 \sin \theta_2 \hat{x}) = 0 \quad (9.62)$$

so although there components of $\mathbf{H}_t$ along the wavevector $k_2$, the dot product is zero.

Power is propagating along the interface, but no power propagates along the $z-$ direction. This is because $\cos \theta_i$ is imaginary, and the average power power propagating in the $z$-direction will have a time dependence of the form $\cos \omega t \sin \omega t$, whose time average is zero.

### 9.0.4 $\pi$ polarization

Note that that here the situation is the same as as for $\sigma$ polarization, except $\mathbf{H}$ plays the role of $\mathbf{E}$, and vice versa.