Chapter 19

Gaussian Beams

We are interested in describing light waves of finite spatial extent, different from plane waves. Gaussian beams are typically generated by laser sources.

19.1 Slowly varying envelope approximation

We begin with the wave equation in an isotropic medium:

$$\nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{19.1}$$

We next assume that the wave is propagating in the $\hat{\mathbf{z}}$ direction, and has the form

$$\mathbf{E} = \Psi(x, y, z)\hat{\mathbf{x}} e^{i(kz - \omega t)} \tag{19.2}$$

Substitution yields

$$\nabla^2 \Psi + 2ik \frac{\partial \Psi}{\partial z} - k^2 \Psi = -\omega^2 \mu\varepsilon \Psi \tag{19.3}$$

We now assume that the amplitude Ψ varies slowly in space; then we get

$$k^2 = \omega^2 \mu\varepsilon \tag{19.4}$$

and since

$$k_o^2 = \omega^2 \mu_o \varepsilon_o \tag{19.5}$$

it follows that

$$n = \frac{k}{k_o} = \sqrt{\frac{\mu}{\mu_o}\frac{\varepsilon}{\varepsilon_o}} \tag{19.6}$$

The equation for the amplitude then becomes

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} + 2ik\frac{\partial\Psi}{\partial z} = 0 \tag{19.7}$$

This is the slowly varying envelope approximation (SVEA). We finally assume that the variation of the amplitude in the z direction is very weak, then we can neglect $\frac{\partial^2\Psi}{\partial z^2}$ compared to the other terms, and we finally get the *paraxial wave equation*

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + 2ik\frac{\partial\Psi}{\partial z} = 0 \tag{19.8}$$

19.2 Gaussian Beam Solution of the Paraxial Wave Equation

We note that we can multiply through by an arbitrary amplitude, and now assume a solution of the form

$$\Psi = e^{-iQ(z)(x^2+y^2)}e^{-iP(z)} \tag{19.9}$$

which implies that the variation of the amplitude in the direction transverse to $\hat{\mathbf{z}}$ is only a function of $x^2 + y^2$. We note that

$$\frac{\partial^2\Psi}{\partial x^2} = -4x^2Q^2\Psi - 2iQ\Psi \tag{19.10}$$

$$\frac{\partial^2\Psi}{\partial y^2} = -4y^2Q^2\Psi - 2iQ\Psi \tag{19.11}$$

$$\frac{\partial\Psi}{\partial z} = -i\frac{\partial P}{\partial z}\Psi - i(x^2+y^2)\frac{\partial Q}{\partial z}\Psi \tag{19.12}$$

and substitution gives

$$-4(x^2+y^2)Q^2\Psi - 4iQ\Psi + 2k\frac{\partial P}{\partial z}\Psi + 2k(x^2+y^2)\frac{\partial Q}{\partial z}\Psi = 0 \tag{19.13}$$

or

$$-2x^2(2Q^2 - k\frac{\partial Q}{\partial z}) - 2y^2(2Q^2 - k\frac{\partial Q}{\partial z}) - 2(2iQ - k\frac{\partial P}{\partial z}) = 0 \tag{19.14}$$

Since the equation must hold for all values of x and y, we must have and

$$2Q^2 - k\frac{\partial Q}{\partial z} = 0 \tag{19.15}$$

and

$$2iQ - k\frac{\partial P}{\partial z} = 0 \tag{19.16}$$

Eq.19.15 is Riccati's equation. It is useful to make a transformation of variables, and let $q = k/2Q$, then

$$\frac{\partial q}{\partial z} = -\frac{k}{2Q^2}\frac{\partial Q}{\partial z} \tag{19.17}$$

and in terms of q, we have

$$\frac{\partial q}{\partial z} = -1 \tag{19.18}$$

and

$$\frac{\partial P}{\partial z} = \frac{i}{q} \tag{19.19}$$

The first of these gives

$$q = iz_o - z \tag{19.20}$$

where we have chosen the constant of integration to be iz_o for convenience, and

$$Q = \frac{k}{2(iz_o - z)}$$

while the second becomes

$$\frac{\partial P}{\partial z} = \frac{i}{iz_o - z} \tag{19.21}$$

and

$$P(z) = -i\ln(iz_o - z) + const. \tag{19.22}$$

It is convenient to choose the constant so that $P(0) = 0$, then

$$P = -i\ln(1 - z/iz_o) \tag{19.23}$$

We remark that the constants of integration can be choosen arbitrarily - for different constants, different solutions to Maxwell's equaitons are obained. We made these particular choices to get the standard Gaussian beam.

We now substitute back into the original expression for Ψ, and

$$\Psi = e^{-i(-i\ln(1-z/iz_o))+\frac{k}{2(iz_o-z)}(x^2+y^2))} \tag{19.24}$$

We note that

$$\ln(1+\frac{iz}{z_o}) = \ln(\sqrt{1+\frac{z^2}{z_o^2}}e^{i\tan^{-1}\frac{z}{z_o}}) = \ln\sqrt{1+\frac{z^2}{z_o^2}} + i\tan^{-1}\frac{z}{z_o} \tag{19.25}$$

and substitution gives

$$\Psi = \frac{1}{\sqrt{1+\frac{z^2}{z_o^2}}}e^{-kz_o\frac{(x^2+y^2)}{2(z^2+z_o^2)}}e^{ikz\frac{(x^2+y^2)}{2(z^2+z_o^2)}}e^{-i\tan^{-1}\frac{z}{z_o}} \tag{19.26}$$

This is the expression for a Gaussian beam. (Note that in the literature, where people use the formalism that the fields are proportional to $e^{i(\omega t-k\cdot r)}$, the sign of k is negative in the exponent of the term before last.)

The conventional notation is

$$E = E_o\frac{w_o}{w(z)}e^{-\frac{r^2}{w^2(z)}}e^{ik\frac{r^2}{2R(z)}}e^{i\phi(z)} \tag{19.27}$$

where

$$w(z)^2 = w_o^2(1+(\frac{z}{z_o})^2) \tag{19.28}$$

and

$$R(z) = z(1+(\frac{z_o}{z})^2) \tag{19.29}$$

and

$$\phi(z) = -\tan^{-1}(\frac{z}{z_o}) \tag{19.30}$$

and

$$w_o^2 = 2z_o/k = z_o\lambda/\pi \tag{19.31}$$

19.3 Properties of Gaussian Beams

The beam is characterized by only one parameter besides the amplitude E_o and the wavelength λ (or equivalently frequency), the quantity w_o (or equivalently z_o).

We note that the amplitude of the beam is Gaussian, that is,

$$E_o \frac{w_o}{w(z)} e^{-\frac{r^2}{w^2(z)}} \tag{19.32}$$

The amplitude is the greatest on axis, with $r = 0$. The field falls to $1/e$ of its maximum value when $r = w$. So w may be thought of as providing a measure of the beam width; that is, the effective beam radius is given by

$$r_{e_1}(z) = w(z) = w_o \sqrt{(1 + (\frac{z}{z_o})^2)} \tag{19.33}$$

The narrowest part of the beam occurs at $z = 0$, here the beam diameter is w_o. The quantity w_o is called the beam waist size.

For large z, the effective radius becomes

$$r_{e_1} \simeq \frac{w_o}{z_o} z \tag{19.34}$$

and we see that the beam diverges with half-angle θ_1, where

$$\tan \theta_1 = \frac{w_o}{z_o} = \frac{\lambda}{\pi w_o} \tag{19.35}$$

This is shown below.

When $z = z_o$, the radius increases by a factor of $\sqrt{2;}$ the quantity z_o is the called the Rayleigh range and $d = 2z_o$ is the confocal parameter, or the depth of focus.

The beam divergence is, approximately,

$$\theta_1 \simeq \frac{w_o}{z_o} = \frac{2}{w_o k} = \frac{\lambda}{\pi w_o} \tag{19.36}$$

and the total divergence is

$$\theta_{1T} = 2\theta_2 = \frac{2\lambda}{\pi w_o} \tag{19.37}$$

In the above, we have defined the beam radius as the value of r where the field falls to $1/e$ of its maximum value.

Many other definitions of beam width are possible.

The intensity is the square of the field, so

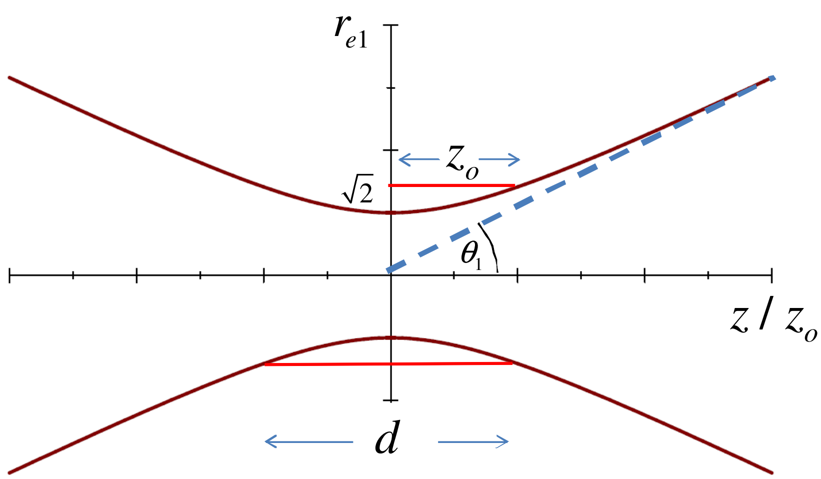

Figure 19.1:

$$I = I_o e^{-\frac{2r^2}{w(z)}} \tag{19.38}$$

The intensity will fall to 1/2 of its maximum value at when $r = r_2$ is such that

$$e^{-\frac{2r_2^2}{w(z)}} = \frac{1}{2} \tag{19.39}$$

that is, when

$$r_2^2 = \frac{w^2(z)}{2} \ln 2 \tag{19.40}$$

and

$$4r_2^2 = w^2(z) 2 \ln 2 \tag{19.41}$$

The beam diameter D (sometimes called *full width at half maximum,* or FWHM) at this point is

$$D = 2r_2 = w(z)\sqrt{2 \ln 2} \tag{19.42}$$

Since $\sqrt{2 \ln 2} = 1.17$,

$$D(z) = 1.17 w(z) \simeq w(z) \tag{19.43}$$

and so we can also regard $w(z)$ as the beam diameter.

Note that

$$r_2 = w(z)\sqrt{\frac{\ln 2}{2}} = r_{e1}\sqrt{\frac{\ln 2}{2}} = 0.59 r_{e1} \tag{19.44}$$

and the beam divergence for FWHM is

$$\theta_2 = 0.59\theta_1 \tag{19.45}$$

so different definitions of beam width and divergence are possible, varying by ˜2. Keeping clear definitions is therefore important.

For simplicity, we will use the θ_1 for the beam divergence.

A Gaussian beam must diverge, otherwise its beam waist is infinite (and we have a plane wave).

The quantity $R(z) = z(1 + (\frac{z_o}{z})^2)$ is a measure of the curvature of the wavefront. At $z = 0$, the phase front is planar

If the wavelength and the divergence of a Gaussian beam is known, both the beam waist and the Rayleigh range can be calculated.For example, if a beam of diameter D is incident on a lens with focal length f, the beam will be focussed. The angle of divergence is $\theta = \tan^{-1} D/2f$.and we have immediately that

$$\frac{w_o}{z_o} = \frac{D}{2f} = \sqrt{\frac{\lambda}{\pi z_o}} \tag{19.46}$$

and

$$z_o = \frac{4\lambda f^2}{\pi D^2} \tag{19.47}$$

and the beam waist is

$$w_o = \sqrt{\frac{z_o\lambda}{\pi}} = \sqrt{\frac{4\lambda^2 f^2}{\pi^2 D^2}} = \frac{2\lambda f}{\pi D} \tag{19.48}$$

A practical problem:

Consider a Gaussian beam, propagating in material 1 with index n_1, incident on a slab of material 2 with index n_2.

In material 1, the beam divergence is given by

$$\tan\theta_1 = \frac{w_{o1}}{z_{o1}} \tag{19.49}$$

and

$$w_{o1}^2 = 2z_{o1}/k_1 = z_{o1}\lambda_o/n_1\pi \tag{19.50}$$

so

$$\tan\theta_1 = \sqrt{\frac{\lambda_o}{\pi n_1 z_{o1}}} \tag{19.51}$$

and, if $\theta_1 << 1$, approximately

$$\sin\theta_1 = \sqrt{\frac{\lambda_o}{\pi n_1 z_{o1}}} \tag{19.52}$$

In material 2, we have similarly

$$\sin\theta_2 = \sqrt{\frac{\lambda_o}{\pi n_2 z_{o2}}} \tag{19.53}$$

If Snell's law holds,

$$n_1\sin\theta_1 = n_2\sin\theta_2 \tag{19.54}$$

and

$$\sqrt{\frac{n_1\lambda_o}{\pi z_{o1}}} = \sqrt{\frac{n_2\lambda_o}{\pi z_{o2}}} \tag{19.55}$$

so

$$\frac{z_{o1}}{n_1} = \frac{z_{o2}}{n_2} \tag{19.56}$$

We also have

$$w_{o1}^2 = 2z_{o1}/k_1 = z_{o1}\lambda_o/n_1\pi \tag{19.57}$$

and

$$w_{o2}^2 = 2z_{o2}/k_2 = z_{o2}\lambda_o/n_1\pi \tag{19.58}$$

and

$$\frac{w_{o1}^2}{w_{o2}^2} = \frac{z_{o1}}{n_1}\frac{n_2}{z_{o2}} = 1 \tag{19.59}$$

and

$$w_{o1} = w_{o2} \tag{19.60}$$

So the waist of the Gaussian beam remains the same, regardless of the medium.