1 Dimensional Analysis

Dimensional analysis is one of the most useful tools of physics

It is a simple method of arriving at hypothetical relationships between physical quantities. Using primarily dimensional arguments, it enables the estimation of material constants and the magnitudes of physical effects. It usually gives correct answer, without having a detailed physical model. Its drawbacks are that on occasion, it gives incorrect/ambiguous results, and that it does not provide a description of the relevant physical processes. .

> **Basic idea:** identify the relevant quantities (this requires some physical insight), and combine these to form a quantity with the appropriate units.

2 Units

Basic dimensional quantities:

	symbol	unit (SI)	unit symbol
length	L	meter	m
time	t	second	s
mass	M	kilogram	kg
electric current	I	Ampere	A
temperature	T	Kelvin	K
luminous intensity			

Everything can be expressed in termsof these(5) quantities**.**

> **Note:** in this course (and in others) all quantities should be given in SI units, in scentific notation, to 3 significant figures.

2.1 Exercise 1.

Express the magnetic permeability $\mu_\circ$ in terms of basic units (m, s, kg, A)

To establish the units of μ_o, think of an equation containing μ, such as the equation for the field inside a solenoid;

$$B = \mu n I \tag{1}$$

Here B is the magnetic flux density, $n =$ no. of turns/length; so $\mu = \frac{B}{nI}$. But what are units of B ? (Tesla, but what is a Tesla in terms of the five basic units?)

Think of an equation with B in it, such as the expression for the Lorentz force,

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{2}$$

and in terms of magnitudes, $F = qvB$ and so $B = \frac{F}{qv} = \frac{Ma}{Itv} = \frac{ML/t^2}{ItL/t} = \frac{M}{It^2}$. Substituting for μ, we finally find

$$\mu_o = \frac{B}{nI} = \frac{M}{I^2t^2nI} = \frac{M}{I^2t^2(1/L)} = \frac{ML}{I^2t^2} = \frac{kg \cdot m}{A^2s^2.} \tag{3}$$

3 Dimensional Analysis - Example 1.

Calculate the speed of waves on the surface of (deep) water.

Want $v = ?(m/s)$.

What does v depend on?

Guess: -acceleration of gravity g

-density of water ρ

-wavelength of waves λ

-?

How can we combine these to have the right units?

try:

$$v = g^\alpha \rho^\beta \lambda^\gamma \tag{4}$$

$$L/t = (\frac{L}{t^2})^\alpha (\frac{M}{L^3})^\beta (L)^\gamma \tag{5}$$

equating exponents: $L \Longrightarrow \quad 1 = \alpha - 3\beta + \gamma$

$t \Longrightarrow \quad -1 = -2\,\alpha$

$M \Longrightarrow \quad 0 = \beta$

so$\beta = 0$, $\alpha = \frac{1}{2}$ and $\gamma = \frac{1}{2}$ and

$$v = \sqrt{g\lambda} \tag{6}$$

This now has physical content; waves have same speed in water or mercury! (Actually, v =$\sqrt{\frac{g\lambda}{2\pi}}$; so our estimate is too large by a factor of$\sim$ 2.)

3.1 Modification of Example 1.

During class, it was suggested that the velocity may depend on the viscosity as well.

Calculate the speed of waves on the surface of (deep) water.

Want $v =?(m/s)$.

What does v depend on?

Guess:
- -acceleration of gravity g
- -density of water ρ
- -wavelength of waves λ
- -viscosity η

How can we combine these to have the right units?

try:

$$v = g^{\alpha}\rho^{\beta}\lambda^{\gamma}\eta^{\delta} \tag{7}$$

$$L/t = \left(\frac{L}{t^2}\right)^{\alpha}\left(\frac{M}{L^3}\right)^{\beta}(L)^{\gamma}\left(\frac{M}{Lt}\right)^{\delta} \tag{8}$$

equating exponents:

$$L \Rightarrow \quad 1 = \alpha - 3\beta + \gamma - \delta$$
$$t \Rightarrow \quad -1 = -2\alpha - \delta$$
$$M \Rightarrow \quad 0 = \beta + \delta$$

Have 3 eqs. in 4 unknowns. Express everything in terms of δ,then $\beta = -\delta,\ \alpha = \frac{1}{2}\ - \frac{\delta}{2}$ and $\gamma = \frac{1}{2}\ - \frac{3}{2}\gamma$ and

$$v = \sqrt{g\lambda}\ \left(\frac{\gamma^2}{g\rho^2\lambda^3}\right)^{\frac{\delta}{2}} \tag{9}$$

The quantity $\frac{\gamma^2}{g\rho^2\lambda^3}$ is dimensionless. From dimensional analysis it is not clear how dimensionless quantities enter an equation.

Physical intuition can help here, however. Suppose $\delta > 0$. Then v would increase with increasing viscosity -this is counter-intuitive, since if we increase viscosity, the speed or particles is decreased, in general, so we would expect v to decrease. Furthermore, $\eta = 0 \Rightarrow\ \ v = 0$. We do NOT expect that viscosity is essential for wave propagation, hence $v \neq 0$ if $\eta = 0$ and so δ cannot be positive.

Suppose then that $\delta < 0$. Then $\eta = 0 \Rightarrow\ v = \infty$, which is unphysical, since we expect v to be proportional to the speed of particles inside the liquid, which must remain finite. (i.e. waves on superfluid He, where $\eta = 0$ do NOT have infinite velocity.) Then we must have $\delta = 0$,and we recover our previous result.

It is possible, however, that the dimensionless group $\frac{\gamma^2}{g\rho^2\lambda^3}$ enters the equation not as a direct product, but instead something like

$$v = \sqrt{g\lambda}\{1 + a\left(\frac{\gamma^2}{g\rho^2\lambda^3}\right)^{\frac{\delta}{2}} + \ldots\}$$

In this case, our result $v = \sqrt{g\lambda}$ is the leading term in an expansion and may be a good approximation, provided higher order terms are small. This is in fact the case for most common liquids.

Dimensionless quantities can have a dramatic effect, however. For example, in critical phenomena, the quantity $(1 - \frac{T}{T_c})$ plays an important role. A correlation length may have a temperature dependence of the form

$$\xi = \frac{\xi_0}{\left(1 - \frac{T}{T_c}\right)^{\alpha}} \tag{10}$$

where T_c is the critical temperature. Dimensional analysis will give us ξ_0. This is a good estimate of ξ so long as T differs appreciably from T_c. As $T \to T_c$, the denominator goes to 0, and ξ_0 is no longer a good estimate of ξ.

4 Dimensional Analysis - Example 2.

How large can a large iceberg be?

Icebergs form by a glacier flowing over land, out over the sea, then breaking off. Most icebergs come from Ellesmere Island in Canada.

What can the typical lengthscale L depend on?

Well, the ice breaks off somehow, so we expect L to depend on the

- the acceleration of gravity g
- the mass density of ice M
- the ultimate strength (tensile stress) Y_c

So guess

$$L = \frac{Y_c}{\rho_M g} \tag{11}$$

But what is Y for ice? Look it up. $Y_c \simeq 10^7 Pa$.

$$L = \frac{10^7}{10^3 \times 10} = 10^3 m \tag{12}$$

This is a reasonable estimate; observed icebergs stick up $\sim 100m$ above sea level.

(An interesting question: how do you estimate Y_c ?)

5 Dimensional Analysis - Example 3.

Estimate the viscosity of water.

To recall definition of viscosity, it is useful to recall an equation involving viscosity - i.e. force F on sheared parallel plates in water.

$$F = \mu A \frac{\triangle v}{\triangle x} \tag{13}$$

where μ is the viscosity, A is the area, $\triangle v$ is the velocity difference of the plates, and $\triangle x$ is their separation. Clearly, the units of μ are $Pa - s (Pa = Pascal = N/m^2 = J/m^3!)$

What can μ depend on? It involves an energy density and a time.The likely energy density is $\rho k T_b$, where T_b is the boiling point of water, and the likely time is the mean free time τ between collisions. Writing $\tau = \frac{l}{v_s}$, where l is a molecular length, and v_s is the speed of sound ($v_s \simeq 1000 m/s$)

$$\begin{aligned} \mu &= \rho k T_b \frac{l}{v_s} = \frac{\rho_m k T_b l}{m v_s} = \frac{\rho_m A_v k T_b l}{M_{mol} v_s} && (14) \\ &= \frac{10^3 \times 8.3 \times 400 \times 3 \times 10^{-10}}{0.018 \times 10^3} \simeq 10^{-4} Pa - s && (15) \end{aligned}$$

The actual viscosity is $\sim 10^{-3} Pa - s$.

6 Dimensional Analysis - Example 4.

Prove Pythagoras' Theorem using dimensional arguments.

It is somewhat surprising that dimensional arguments can be used to derive (prove!) a relation between quantities having the same units.

Proof:

Let a right triange have sides a,b,c where c is the hypotenuse, and α is the angle adjacent to side c, and β is the angle adjacent to side b.

To specify the triangle uniquely, we only need to specify one side and one angle (either α or β).

So we must be able to write an expression for the area A in terms of c and α . That is,

$$A = f(c, \alpha) \tag{16}$$

where f is some function of the arguments c and α. But for dimensional reasons, we must have

$$A = c^2 g(\alpha)$$

where g is some (dimensionless) function of its argument α. Similarly, we must have $A = c^2 g(\beta)\, ,\, so\ g(\alpha) = g(\beta)\, .$

The original triangle can be viewed as being made up of two triangles; separated by a normal line from the hypotenuse to the apex with the right-angle. The areas of each of these are $A_a = a^2 g\ (\alpha)\ and\ A_b = b^2 g(\beta)\, .$ The additivity of areas

$$A = A_a + A_b$$

gives at once

$$c^2 = a^2 + b^2 \tag{17}$$