Chapter 17

Light Propagation in Cholesterics

We consider the propagation of light in liquid crystals. Starting with Maxwell's equations,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{17.1}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \tag{17.2}$$

and letting $\mathbf{B} = \mu_o \mathbf{H}$ and $\mathbf{D} = \varepsilon_o \varepsilon_r \mathbf{E}$ gives

$$\nabla \times \nabla \times \mathbf{E} = -\mu_o \varepsilon_o \varepsilon_r \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{17.3}$$

or

$$\nabla^2 \mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) = \frac{\varepsilon_r}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{17.4}$$

We now let

$$\hat{\boldsymbol{\eta}} = \cos qz \hat{\imath} + \sin qz \mathbf{j} \tag{17.5}$$

$$\hat{\boldsymbol{\phi}} = -\sin qz \mathbf{i} + \cos qz \hat{\jmath} \tag{17.6}$$

and note that in the $\hat{\boldsymbol{\eta}}$, $\hat{\phi}$ frame

$$\varepsilon_r = \begin{bmatrix} \varepsilon_{||} & 0 \\ 0 & \varepsilon_{\perp} \end{bmatrix} \tag{17.7}$$

is diagonal. That is, the axis of the cholesteric helix is along the z-direction, and the director traces out a right-handed helix.(for definition of handedness and helicity, see the Appendix.)

We let the electric field be of the form

$$\mathbf{E} = \mathbf{E}(z)e^{i(kz-\omega t)} \tag{17.8}$$

and assume that $\mathbf{E}(z)$ is in the $x-y$ plane. We note that $\nabla \cdot \mathbf{E} = 0$, and Eq. 17.4 gives

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} = -\frac{\omega^2}{c^2}\varepsilon_r \mathbf{E} \tag{17.9}$$

Substitution of Eq. 17.8 gives

$$\mathbf{E}''(z) + 2ik\mathbf{E}'(z) - k^2\mathbf{E}(z) = -\frac{\omega^2}{c^2}\varepsilon_r \mathbf{E}(z) \tag{17.10}$$

We now write

$$\mathbf{E}(z) = E_{||}\hat{\boldsymbol{\eta}} + E_{\perp}\hat{\boldsymbol{\phi}} \tag{17.11}$$

where $E_{||}$ and $E_{\perp}$ are constants. Since $\hat{\boldsymbol{\eta}}' = q\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{\phi}}' = -q\hat{\boldsymbol{\eta}}$, substitution of Eq. 17.11 into Eq.17.10 gives

$$(-q^2 + \frac{\omega^2}{c^2}\varepsilon_{||} - k^2)E_{||} = 2ikqE_{\perp} \tag{17.12}$$

and

$$(-q^2 + \frac{\omega^2}{c^2}\varepsilon_{\perp} - k^2)E_{||} = -2ikqE_{||} \tag{17.13}$$

Now $q = 2\pi/p$, where p is the cholesteric pitch, $k = 2\pi n/\lambda_o$ where λ_o is the wavelength of light in free space, n is the effective refractive index ($\lambda = \lambda_o/n$ is the wavelength in the $\hat{\boldsymbol{\eta}}$, $\hat{\boldsymbol{\phi}}$ frame). We note that $\omega/c = 2\pi/\lambda_o$, and let $\alpha = \lambda_o/p$. Then Eqs. 17.12 and 17.13 become

$$(\alpha^2 - \varepsilon_{||} + n^2)E_{||} = -2in\alpha E_{\perp} \tag{17.14}$$

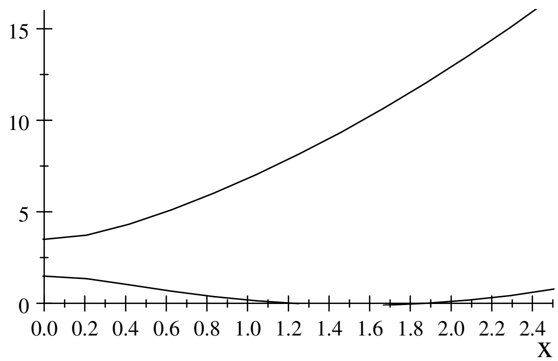
and

$$(\alpha^2 - \varepsilon_{\perp} + n^2)E_{\perp} = 2in\alpha E_{||} \tag{17.15}$$

These give, after some algebra,

$$n^2 = \overline{\varepsilon} + \alpha^2 \pm \sqrt{\delta^2 + 4\overline{\varepsilon}\alpha^2} \tag{17.16}$$

where $\overline{\varepsilon} = (\varepsilon_{||} + \varepsilon_{\perp})/2$ is the average dielectric constant, and $\delta = (\varepsilon_{||} - \varepsilon_{\perp})/2$ is the dielectric anisotropy.



n^2 versus $\alpha = \lambda_o/p$ for $\varepsilon_{||} = 3$, and $\varepsilon_\perp = 2$.

We denote the value of n from Eq. 17.16 as n_+ and with the negative as n_-. If $\alpha \approx 0$,

$$n^2 = \overline{\varepsilon} + \alpha^2 \pm (\delta + 2\overline{\varepsilon}\alpha^2/\delta) \tag{17.17}$$

and

$$n_+^2 = \varepsilon_{||} + \alpha^2(1 + 2\overline{\varepsilon}/\delta) \tag{17.18}$$

$$n_-^2 = \varepsilon_\perp + \alpha^2(1 - 2\overline{\varepsilon}/\delta) \tag{17.19}$$

Thus, for $\alpha = 0$, the two refractive indices are $n_+ = \sqrt{\varepsilon_{||}}$ and $n_- = \sqrt{\varepsilon_\perp}$, and from Eq.17.14 and 17.15 we see that the eigenmodes are two plane polarized waves, one parallel to $\hat{\boldsymbol{\eta}}$ (the director) and the other to $\hat{\boldsymbol{\phi}}$. If $\alpha \neq 0$ but small (the cholesteric pitch is long compared to the wavelength), then the eigenmodes are given by

$$(1 + \frac{\overline{\varepsilon}}{\delta})E_{||}^+ \simeq -i\frac{\sqrt{\varepsilon_{||}}}{\alpha}E_\perp^+ \tag{17.20}$$

$$(1 - \overline{\varepsilon}/\delta)E_\perp^- \simeq i\frac{\sqrt{\varepsilon_\perp}}{\alpha}E_{||}^- \tag{17.21}$$

or

$$\sqrt{\varepsilon_{||}}E_{||}^+ \simeq -i\frac{\delta}{\alpha}E_\perp^+ \tag{17.22}$$

$$\sqrt{\varepsilon_\perp}E_\perp^- \simeq -i\frac{\delta}{\alpha}E_{||}^- \tag{17.23}$$

that is, the two eigenmodes ar two elliptically polarized waves. Explicitly, substituting into Eq. 17.11 gives

$$\mathbf{E}^+ = E_{||}^+ \left[\hat{\boldsymbol{\eta}}\cos(kz - \omega t) - \frac{\alpha\sqrt{\varepsilon_{||}}}{\delta}\hat{\boldsymbol{\phi}}\sin(kz - \omega t)\right] \tag{17.24}$$

which is left-circularly polarized (with positive helicity), opposite to the handedness of the cholesteric, which is right-handed; and

$$\mathbf{E}^{-} = E_{\perp}^{-}\left[-\frac{\alpha\sqrt{\varepsilon_{\perp}}}{\delta}\hat{\boldsymbol{\eta}}\sin(kz-\omega t)+\hat{\boldsymbol{\phi}}\cos(kz-\omega t)\right] \tag{17.25}$$

which is right-circularly polarized (with negative helicity), having the same handedness as the cholesteric.

It is interesting to look at the behavior in the vicinity of the region where n^2 becomes negative. Imaginary n means that there is no propagation; hence expect strong reflection of one mode here. The end points of this reflection band are the points where $n_{-}^{2} = 0$. Here

$$n_{-}^{2} = \bar{\varepsilon} + \alpha^2 - \sqrt{\delta^2 + 4\bar{\varepsilon}\alpha^2} = 0 \tag{17.26}$$

This gives $\alpha = \sqrt{\varepsilon_{\perp}}$ and $\alpha = \sqrt{\varepsilon_{||}}$; that is, reflection occurs when

$$p\sqrt{\varepsilon_{\perp}} < \lambda_o < p\sqrt{\varepsilon_{||}} \tag{17.27}$$

or

$$pn_o < \lambda_o < pn_e \tag{17.28}$$

We consider the fields at the reflection band edges.

If $\alpha = \sqrt{\varepsilon_{\perp}}$, then $n_{-} = 0$. Eq.17.15 is automatically satisfied; Eq. 17.14 $\Longrightarrow E_{||}^{-} = 0$. Since $k = \frac{2\pi n}{\lambda_o} = 0$, the solution is

$$\mathbf{E}^{-} = E_{\perp}^{-}\hat{\boldsymbol{\phi}}e^{-i\omega t} \tag{17.29}$$

Here the field is everywhere parallel to $\hat{\boldsymbol{\phi}}$ and oscillating in time; this is a non-propagating mode. For the same value $\alpha = \sqrt{\varepsilon_{\perp}}$, from Eq.17.16, $n_{+}^{2} = 2(\bar{\varepsilon} + \alpha^2)$ and $n_{+}^{2} = (\varepsilon_{||} + 3\varepsilon_{\perp})$. Then Eq. 17.14 gives

$$E_{\perp}^{+} = 2i\sqrt{\frac{\varepsilon_{\perp}}{\varepsilon_{||}+3\varepsilon_{\perp}}}E_{||}^{+} \tag{17.30}$$

and the solution is

$$\mathbf{E}^{+} = E_{||}^{+}\left[\hat{\boldsymbol{\eta}}\cos(kz-\omega t) - 2\sqrt{\frac{\varepsilon_{\perp}}{\varepsilon_{||}+3\varepsilon_{\perp}}}\hat{\boldsymbol{\phi}}\sin(kz-\omega t)\right] \tag{17.31}$$

which is again left-circularly polarized (with positive helicity), opposite to the handedness of the cholesteric.

Similarly, if $\alpha = \sqrt{\varepsilon_{||}}$, then $n_- = 0$. Eq. 17.14 is automatically satisfied; Eq. 17.15 $\Longrightarrow E_\perp^- = 0$. Since $k = \frac{2\pi n}{\lambda_o} = 0$, the solution is

$$\mathbf{E}^- = E_{||}^- \hat{\boldsymbol{\eta}} e^{i\omega t} \tag{17.32}$$

Here the field is everywhere parallel to $\hat{\boldsymbol{\eta}}$ and oscillating in time; this is a non-propagating mode. For the same value $\alpha = \sqrt{\varepsilon_{||}}$, from Eq.17.16, $n_+^2 = 2(\bar{\varepsilon} + \alpha^2)$ and $n_+^2 = 3\varepsilon_{||} + \varepsilon_\perp$.Then Eq. 17.14 gives

$$E_{||}^+ = -2i\sqrt{\frac{\varepsilon_{||}}{3\varepsilon_{||} + \varepsilon_\perp}} E_\perp^+ \tag{17.33}$$

and the solution is

$$\mathbf{E}^+ = E_\perp^+ \left[2\sqrt{\frac{\varepsilon_\perp}{3\varepsilon_{||} + \varepsilon_\perp}} \hat{\boldsymbol{\eta}} \sin(kz - \omega t) + \hat{\boldsymbol{\phi}} \cos(kz - \omega t) \right] \tag{17.34}$$

which is again left-circularly polarized (with positive helicity), opposite to the handedness of the cholesteric.

Finally, it is interesting to ask what happens when the pitch gets much shorter than the wavelength of light. From Eq. 17.16, if $\alpha >> \bar{\varepsilon}$,

$$n^2 \simeq (\sqrt{\bar{\varepsilon}} \pm \alpha)^2 \tag{17.35}$$

The relevant roots are

$$n_+ = \sqrt{\bar{\varepsilon}} + \alpha \tag{17.36}$$

and from Eq.17.14,

$$E_{||} \simeq i E_\perp \tag{17.37}$$

so

$$\mathbf{E} = E_\perp \left[\hat{\boldsymbol{\eta}} \sin(kz - \omega t) + \hat{\boldsymbol{\phi}} \cos(kz - \omega t) \right] \tag{17.38}$$

which is left-circularly polarized, opposite to the handedness of the cholesteric, with wavevector

$$k_+ = \frac{2\pi\sqrt{\bar{\varepsilon}}}{\lambda_o} + \frac{2\pi\alpha}{\lambda_o} = k_A + q \tag{17.39}$$

Now substituting for k and writing $\mathbf{E}$ in the lab-frame, we get

$$\begin{aligned} \mathbf{E}^+ = E_\perp \Big[& -\hat{\imath}\{(\cos qz \sin(-qz + k_A z - \omega t) \\ & + \sin qz \cos(-qz + k_A z - \omega t)\} \\ & + \hat{\jmath}\{\cos qz \cos(-qz + k_A z - \omega t) \\ & - \sin qz \sin(-qz + k_A z - \omega t)\} \Big] \end{aligned} \tag{17.40}$$

or

$$\mathbf{E}^{+}=E_{\perp}\left[-\hat{\imath}\sin(k_A z-\omega t)+\hat{\jmath}\cos(k_A z-\omega t)\}\right] \tag{17.41}$$

This is circularly polarized light, propagating with speed $c/\sqrt{\bar{\varepsilon}}$, independent of the pitch of the cholesteric.

Similarly,

$$n_{-}=\sqrt{\bar{\varepsilon}}-\alpha \tag{17.42}$$

Note here that $n_{-}<0$. Fom Eq.17.15,

$$E_{\perp}=iE_{||} \tag{17.43}$$

and

$$\mathbf{E}^{-}=E_{||}\left[\hat{\boldsymbol{\eta}}\cos(kz-\omega t)-\hat{\boldsymbol{\phi}}\sin(kz-\omega t)\right] \tag{17.44}$$

which is left circularly polarized, since $k<0$;

$$k_{-}=\frac{2\pi\sqrt{\bar{\varepsilon}}}{\lambda_o}-\frac{2\pi\alpha}{\lambda_o}=k_A-q \tag{17.45}$$

and substituting for k_{-} and writing $\mathbf{E}$ in the lab-frame, we get

$$\begin{aligned}\mathbf{E}^{-} = E_{||}\Big[& -\hat{\imath}\{(\cos qz\cos(qz+k_A z-\omega t) \\ & +\sin qz\sin(qz+k_A z-\omega t)\} \\ & +\hat{\jmath}\{\sin qz\cos(qz+k_A z-\omega t) \\ & -\cos qz\sin(qz+k_A z-\omega t)\}\Big]\end{aligned} \tag{17.46}$$

or

$$\mathbf{E}^{-}=E_{||}\left[\hat{\imath}\cos(k_A z-\omega t)-\hat{\jmath}\sin(k_A z-\omega t)\right] \tag{17.47}$$

The normal modes in Eqs. 17.41 and 17.47 are left- and right-circularly polarized waves in the lab frame, travelling with speed $c/\sqrt{\bar{\varepsilon}}$, independent of the pitch of the cholesteric. They may be combined to give plane polarized light. Thus, for $\alpha >> 1$, the light essentially sees an isotropic medium, with dieletric constant $\bar{\varepsilon}$.

17.1 Summary

In summary, the normal modes are two elliptically polarized waves whose principal axes remain aligned with the principal axes of the dielectric tensor.

In the case of long pitch, ($\alpha \simeq 0$), the normal modes are essentially plane polarized waves, whose polarization follows the principal axes of the dielectric tensor. The n_+ mode corresponds to $\mathbf{E}$ along the director, and the n_- mode to $\mathbf{E}$ perpendicular to the director.

In general, the n_+ mode has the opposite handedness as the cholesteric. This mode becomes more and more circularly polarized as the pitch decreases (α increases), and it always propagates.

For small α, the n_- mode has the same handedness as the cholesteric. As the pitch decreases, this mode first becomes more and more circularly polarized, then less circularly polarized, and becomes plane polarized (in the rotating frame) when $\lambda_o = p\sqrt{\varepsilon_\perp}$. Here $n_- = 0$; the $\mathbf{E}$-field resembles a standing wave; its direction remains everywhere perpendicular to the director, and its magnitude varies in time but not in space. This may be thougth of as two counterpropagating circularly polarized waves, both with the same handedness as the cholesteric. The Poynting vector is zero. In the region $p\sqrt{\varepsilon_\perp} < \lambda_o < p\sqrt{\varepsilon_{||}}$, there is no propagation of the n_- mode, since n_- is imaginary; here the field decays exponentially in the z-direction. The n_+ mode, which is nearly circularly polarized, propagates, however. When $\lambda_o = p\sqrt{\varepsilon_{||}}$, again $n_- = 0$, and the $\mathbf{E}$ field is everywhere parallel to the director, its magnitude oscillates in time but not in space. This again may be thought of as two counterpropagating circularly polarized waves, with the same handedness as the cholesteric. As the pitch becomes shorter, the n_-mode again propagates, but now with opposite handedness as the cholesteric, as an elliptically polarized wave, becoming more and more circularly polarized with decreasing p. For large α, the material appears isotropic with effective index $\sqrt{\bar{\varepsilon}}$.

17.2 Appendix

We discuss handedness and helicity.

Handedness is associated with space alone, while helicity is with space and time. Helicity is associated with the sign of angular momentum projected onto the propagation direction.

Consider a vector field given by

$$\hat{\mathbf{E}} = \hat{\imath}\cos(kz - \omega t) + \hat{\jmath}\sin(kz - \omega t) \tag{17.48}$$

At a fixed time (say $t = 0$), as function of position z, $\hat{\mathbf{E}}$, given by Eq. 17.48, traces out a *right handed* helix, similar to a right handed screw. This is *right polarized light*. If you are looking so that light is coming towards you, the helix is rotating counteclockwise as the spatial coordinate z increases.

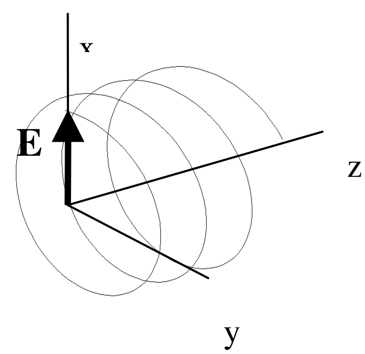


Fig. 1. Right handed polarization. For light coming towards observer, the helix is rotating counteclockwise as function of position.

At a fixed position (say $z = 0$), as function of time t, $\hat{\mathbf{E}}$, given by Eq. 17.48 traces out a left handed helix, similar to a left-handed screw. (Note that the direction of increasing time is in the direction of wave propagation) This is light with *negative helicity* (left handed helicity). If you are looking so that light is coming towards you, the helix is rotating clockwise as the temporal coordinate t increases. The angular momentum vector is pointing away from you.

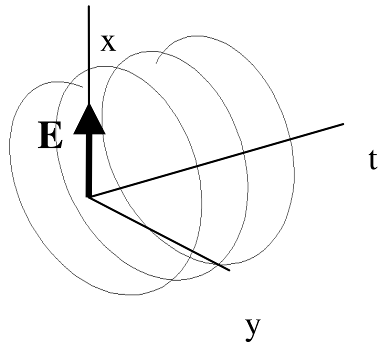


Fig. 2. Negative helicity. For light coming towards observer, the helix is rotating clockwise as function of time.

Right circularly polarized light has negative (left handed) helicity.

Left circularly polarized light has positive (right handed) helicity.

Handedness is associated with space alone, while helicity is with space and time. For travelling waves, the two have opposite handedness; this can cause confusion in the literature.

Light has both handedness and helicity; the cholesteric material only handedness.