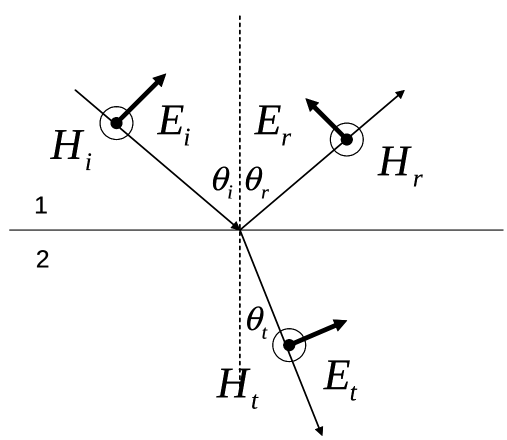
Chapter 9

Fresnel Reflection

9.0.1 π polarization:



We have from the law of reflection that

$$\theta_i = \theta_r \tag{9.1}$$

We have from Snell's law that

$$n_1 \sin \theta_i = n_2 \sin \theta_t \tag{9.2}$$

Boundary conditions for the tangential components of **H** give us

$$H_i + H_r = H_t \tag{9.3}$$

or

$$\frac{E_i}{Z_1} + \frac{E_r}{Z_1} = \frac{E_t}{Z_2} \tag{9.4}$$

or

$$E_t = \frac{Z_2}{Z_1}(E_i + E_r)$$

and

$$E_i \cos\theta_i - E_r \cos\theta_r = E_t \cos\theta_t \tag{9.5}$$

and

$$E_i \cos\theta_i - E_r \cos\theta_i = \frac{Z_2}{Z_1}(E_i + E_r)\cos\theta_t \tag{9.6}$$

and defining

$$r = \frac{E_r}{E_i} \tag{9.7}$$

, we have

$$\cos\theta_i - r\cos\theta_i = \frac{Z_2}{Z_1}(1 + r_r)\cos\theta_t \tag{9.8}$$

and

$$cas\theta_i - \frac{Z_2}{Z_1}\cos\theta_t = r(\cos\theta_i + \frac{Z_2}{Z_1}\cos\theta_t) \tag{9.9}$$

and finally

$$r = \frac{Z_1\cos\theta_i - Z_2\cos\theta_t}{Z_1\cos\theta_i + Z_2\cos\theta_t} \tag{9.10}$$

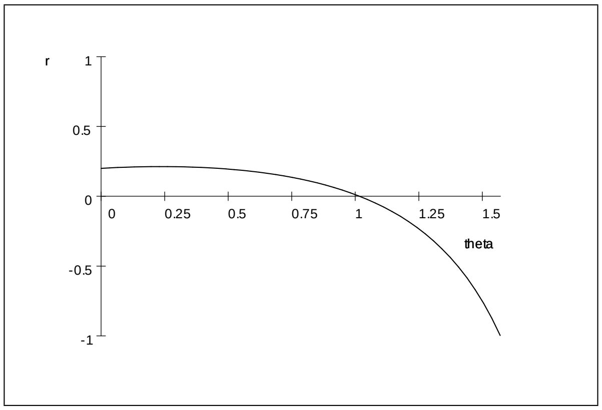
If the material is non-magnetic, then

$$r = \frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t} \tag{9.11}$$

and writing θ_t explicitly, we get

$$r = \frac{n_2\cos\theta_i - n_1\sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\theta_i}}{n_2\cos\theta_i + n_1\sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\theta_i}} \tag{9.12}$$

Plotting this for glass, with $n_1 = 1$, $n_2 = 1.5$, gives



We note that r goes to zero at the Brewster angle; here $\tan\theta_i = \frac{n_2}{n_1}$.

We also define the transmission coefficient

$$t = \frac{E_t}{E_i} \tag{9.13}$$

which, since

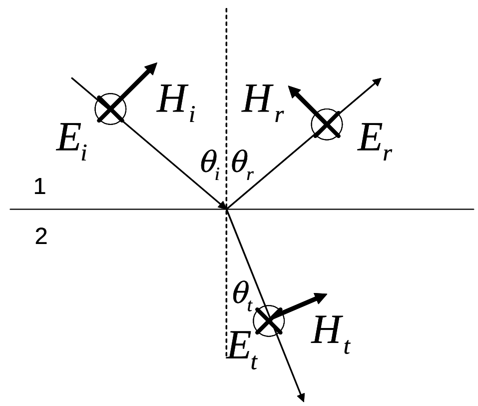
$$E_t = \frac{Z_2}{Z_1}(E_i + E_r) \tag{9.14}$$

becomes

$$t = \frac{Z_2}{Z_1}(1+r) = \frac{Z_2}{Z_1}(1+\frac{Z_1\cos\theta_i - Z_2\cos\theta_t}{Z_1\cos\theta_i + Z_2\cos\theta_t}) = \frac{2Z_2\cos\theta_i}{Z_1\cos\theta_i + Z_2\cos\theta_t} = \frac{2n_1\cos\theta_i}{n_2\cos\theta_i + n_1\cos\theta_t} \tag{9.15}$$

9.0.2 σ Polarization

For σ polarization, we have



and here

$$E_i + E_r = E_t \tag{9.16}$$

or

$$\frac{H_i}{Z_1^{-1}} + \frac{Hr}{Z_1^{-1}} = \frac{H_t}{Z_2^{-1}} \tag{9.17}$$

or

$$H_t = \frac{Z_2^{-1}}{Z_1^{-1}}(H_i + H_r)$$

and

$$H_i \cos\theta_i - H_r \cos\theta_r = H_t \cos\theta_t \tag{9.18}$$

and

$$H_i \cos\theta_i - H_r \cos\theta_i = \frac{Z_2^{-1}}{Z_1^{-1}}(H_i + H_r)\cos\theta_t \tag{9.19}$$

and defining

$$r = \frac{E_r}{E_i} = \frac{H_r}{H_i} \tag{9.20}$$

, we have

$$\cos\theta_i - r\cos\theta_i = \frac{Z_2^{-1}}{Z_1^{-1}}(1 + r_r)\cos\theta_t \tag{9.21}$$

and

$$cas\theta_i - \frac{Z_2^{-1}}{Z_1^{-1}}\cos\theta_t = r(\cos\theta_i + \frac{Z_2^{-1}}{Z_1^{-1}}\cos\theta_t) \tag{9.22}$$

and finally

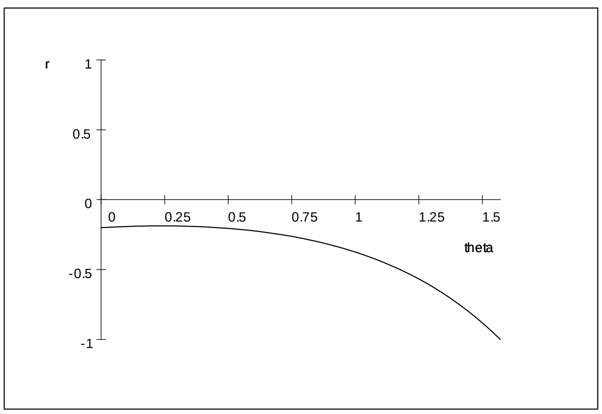
$$r = \frac{Z_2 \cos\theta_i - Z_1 \cos\theta_t}{Z_2 \cos\theta_i + Z_1 \cos\theta_t} \tag{9.23}$$

If the material is non-magnetic, then

$$r = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t} \tag{9.24}$$

and writing θ_t explicitly, we get

$$r = \frac{n_1 \cos\theta_i - n_2\sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\theta_i}}{n_1 \cos\theta_i + n_2\sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\theta_i}} \tag{9.25}$$



The transmittance is

$$t = \frac{E_t}{E_i} \tag{9.26}$$

and since

$$E_i + E_r = E_t \tag{9.27}$$

we have

$$\begin{aligned} t &= 1 + r = 1 + \frac{Z_2 \cos\theta_i - Z_1 \cos\theta_t}{Z_2 \cos\theta_i + Z_1 \cos\theta_t} = \frac{2Z_2 \cos\theta_i}{Z_2 \cos\theta_i + Z_1 \cos\theta_t} \\ &= \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t} \end{aligned}$$

Summary

At an interface between two materials, we have, for π polarization

$$r_{12}^{\pi} = \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_2 \cos\theta_1 + n_1 \cos\theta_2} \tag{9.28}$$

and

$$t_{12}^{\pi} = \frac{2n_1 \cos\theta_1}{n_2 \cos\theta_1 + n_1 \cos\theta_2} \tag{9.29}$$

while for σ polarization, we have

$$r_{12}^{\sigma} = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \tag{9.30}$$

and

$$t_{12}^{\sigma} = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \tag{9.31}$$

9.0.3 Total Internal Reflection (TIR)

σ polarization

Consider the reflection coefficient for σ polarization.

$$r_{12}^{\sigma} = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \tag{9.32}$$

Using Snell's Law, we can write

$$\sin\theta_2 = \frac{n_1}{n_2} \sin\theta_1$$

$$\cos\theta_2 = \sqrt{1 - \frac{n_1^2}{n_2^2} \sin^2\theta_1} \tag{9.33}$$

$$r_{12}^{\sigma} = \frac{n_1 \cos\theta_1 - n_2 \sqrt{1 - \frac{n_1^2}{n_2^2} \sin^2\theta_1}}{n_1 \cos\theta_1 + n_2 \sqrt{1 - \frac{n_1^2}{n_2^2} \sin^2\theta_1}} \tag{9.34}$$

Now suppose that $n_1 > n_2$. In this case, if

$$\sin\theta_1 > \frac{n_2}{n_1} \tag{9.35}$$

we have

$$\cos\theta_2 = i\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1} \tag{9.36}$$

and $\cos\theta_2$ is imaginary (but $\sin\theta_t$ remains real), and r_{12}^σ is complex. It is useful to define the critical angle for TIR as

$$\sin\theta_c = \frac{n_2}{n_2} \tag{9.37}$$

or

$$\theta_c = \sin^{-1}\frac{n_2}{n_1} \tag{9.38}$$

Now we can write

$$r_{12}^\sigma = \frac{n_1\cos\theta_1 - n_2 i\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1}}{n_1\cos\theta_1 + n_2 i\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1}} = \frac{a - ib}{a + ib} \tag{9.39}$$

or

$$r_{12}^\sigma = |r_{12}^\sigma| e^{i\phi} \tag{9.40}$$

We write

$$r_{12}^\sigma = \frac{a - ib}{a + ib} \tag{9.41}$$

and note that

$$|r_{12}^\sigma| = \sqrt{r_{12}^\sigma r_{12}^{\sigma *}} = 1 \tag{9.42}$$

So the magnitude of the reflection coefficient is 1. Writing

$$r_{12}^\sigma = \frac{a - ib}{a + ib} = \frac{(a - ib)^2}{a^2 + b^2} = \frac{a^2 - b^2}{a^2 + b^2} - \frac{2iab}{a^2 + b^2} \tag{9.43}$$

the phase ϕ can be conveniently written as

$$\tan\phi = -\frac{2ab}{a^2 - b^2} = -\frac{2n_1\cos\theta_1 n_2\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1}}{n_1^2\cos^2\theta_1 - n_2^2(\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1)} = -\frac{2n_1\cos\theta_1\sqrt{n_1^2\sin^2\theta_1 - n_2^2}}{n_1^2(\cos^2\theta_1 - \sin^2\theta_1) + n_2^2}$$

The reflected wave has the same amplitude as the incident wave, but there is an angle-dependent phase shift. The reflection coefficient for the intensity is

$$R^\sigma = r_{12}^\sigma r_{12}^{\sigma *} = 1 \tag{9.44}$$

All of the incident power is reflected - this is total internal reflection. No power is transmitted.

The transmission coefficient is

$$t_{12}^{\sigma} = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2} = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 i\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1}} \tag{9.45}$$

a complex number, which equals 2 at the critical angle of incidence θ_c.

It is interesting to look at the wavevector $\mathbf{k}_2$. From Maxwell'e equations we have (assuming isotropy and nonmagnetic materials) that

$$k_2^2 = \omega^2 \mu_o \varepsilon \tag{9.46}$$

or

$$k_2 = \frac{\omega n_2}{c} \tag{9.47}$$

The in-plane component of $\mathbf{k}_2$ (in the interface)

$$k_{2x} = k_2 \sin\theta_2 = k_1 \sin\theta_1 \tag{9.48}$$

and, since $\sin\theta_2$ is real, the in-plane component of the wave-vector $\mathbf{k}_2$ is real. Furthermore, θ_2 is such that Snell's law is satisfied;

$$k_2 \sin\theta_2 = k_1 \sin\theta_1 \tag{9.49}$$

We also have for the normal component k_{2z} of $\mathbf{k}_2$ that

$$k_{2z} = k_2 \cos\theta_2 \tag{9.50}$$

but this is imaginary, since

$$k_{2z} = k_2 i\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1} \tag{9.51}$$

so we have an imaginary normal component of the wave vector. We can also write this as

$$k_{2z} = i\frac{n_1}{n_2}k_2\sqrt{\sin^2\theta_1 - \sin^2\theta_c} = ik_1\sqrt{\sin^2\theta_1 - \sin^2\theta_c}$$

In essence, what happens is the following. As the angle of incidence $\theta_1$increases, the angle of transmission θ_2 grows. At the critical angle,when

$\theta_1 = \theta_c$, $\mathbf{k}_2$ is in the plane of the interface. As the angle of incidence increases further, an imaginary component of $\mathbf{k}_2$ appears, normal to the interface. This increases the length of $\mathbf{k}_2$ sufficiently so that its projection on the interface can match that of the incident wave.

Note that

$$k_2^2 = \mathbf{k}_2 \cdot \mathbf{k}_2 = k_{2x}^2 - |k_{2z}^2|^2 \tag{9.52}$$

So, indeed, having an imaginary component allows k_2 to be longer. Substitution gives

$$k_1^2 \sin^2\theta_1 - k_2^2(\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1) = k_2^2 = \frac{\omega n_2}{c} \tag{9.53}$$

as expected.

The transmitted electric field at the interface is

$$\mathbf{E}_t = \mathbf{E}_i t_{12} \tag{9.54}$$

so it is just the incident field multiplied by a complex amplitude. Explicitly, we have

$$\mathbf{E}_t = E_{ot}\hat{\mathbf{y}}e^{i(\mathbf{k}_2\cdot\mathbf{r}-\boldsymbol{\omega}\mathbf{t})} = E_{ot}\hat{\mathbf{y}}\mathbf{e}^{-k_{2z}z}e^{i(k_2 x-\boldsymbol{\omega}\mathbf{t})} \tag{9.55}$$

so we have a real wave propagating along the interface, with wave vector

$$k_{2x} = k_2 \sin\theta_2 = k_1 \sin\theta_1 \tag{9.56}$$

and an exponentially decaying envelope normal to the interface with decay length

$$\zeta = \frac{1}{k_{2z}} = \frac{1}{k_1\sqrt{\sin^2\theta_1 - \sin^2\theta_c}} = \frac{\lambda_o}{2\pi n_1\sqrt{\sin^2\theta_1 - \sin^2\theta_c}} \tag{9.57}$$

The $\mathbf{H}$ field is given, as usual, by

$$\mathbf{H}_t = \frac{1}{\omega\mu_o}\mathbf{k}_2 \times \mathbf{E}_t \tag{9.58}$$

or

$$\mathbf{H}_t = \frac{1}{Z_2}\hat{\mathbf{k}}_2 \times \mathbf{E}_t \tag{9.59}$$

and

$$\mathbf{H}_t = \frac{1}{Z_2}(\sin\theta_2\hat{\mathbf{x}} + \cos\theta_2\hat{\mathbf{z}}) \times E_t\hat{\mathbf{y}} = \frac{1}{Z_2}E_t\sin\theta_2\hat{\mathbf{z}} - \frac{1}{Z_2}E_t\cos\theta_2\hat{\mathbf{x}} \tag{9.60}$$

so $\mathbf{H}_t$ has a real component along the $z-$ direction, and an imaginary component along the x direction.

Is this still a transverse wave?

Is the relation

$$\nabla \cdot \mathbf{H} = \mathbf{k}_2 \cdot \mathbf{H}_t = 0 \tag{9.61}$$

satisfied? Substitution gives

$$\mathbf{k}_2 \cdot \mathbf{H}_t = (\frac{1}{Z_2} E_t \sin\theta_2 \hat{\mathbf{z}} - \frac{1}{Z_2} E_t \cos\theta_2 \hat{\mathbf{x}}) \cdot (k_2 \cos\theta_2 \hat{\mathbf{z}} + k_2 \sin\theta_2 \hat{\mathbf{x}}) = 0 \tag{9.62}$$

so although there componentsof $\mathbf{H}_t$ along the wavevector $\mathbf{k}_2$, the dot product

is zero.

Power is propagating along the interface, but no power propagates along the $z-$direction. This is because $\cos\theta_t$ is imaginary, and the average power power propagating in the z-direction will have a time dependence of the form $\cos\omega t \sin\omega t$, whose time average is zero.

9.0.4 π polarization

Note that that here the sitatuation is the same as as for σ polarization, except $\mathbf{H}$ plays the role of $\mathbf{E}$, and vice versa.