Chapter 8

Inhomogeneous Solutions

We have shown that, in vacuum,Maxwell's equations reduce to the wave equation

$$\nabla^2 \mathbf{E} = \mu_o \varepsilon_o \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{8.1}$$

which needs to be solved subject to

$$\nabla \cdot \mathbf{E} = \mathbf{0} \tag{8.2}$$

Such a solution is

$$\mathbf{E} = \mathbf{E}_o \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \tag{8.3}$$

where $\mathbf{E}_o \cdot k = 0$; substitution into the wave equation gives

$$-k^2 \mathbf{E} = -\mu_o \varepsilon_o \omega^2 \mathbf{E} \tag{8.4}$$

or

$$k^2 = \mu_o \varepsilon_o \omega^2 \tag{8.5}$$

which determines the magnitude of k (that is, the wavelength, since $k = 2\pi/\lambda$). This is called the 'dispersion relation', relating the wave vector to the frequecy. From this, we can get the velocity;

$$v = \frac{k}{\omega} = \frac{1}{\sqrt{\varepsilon_o \mu_o}} = c \tag{8.6}$$

Once $\mathbf{E}$ is known, the other fields can be easily obtained. For example,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{8.7}$$

gives

$$-\mathbf{k} \times \mathbf{E}_o \sin(\mathbf{k} \cdot \mathbf{r} - \omega t) = -\mu_o \frac{\partial \mathbf{H}}{\partial t} \tag{8.8}$$

or

$$\mathbf{H} = \frac{1}{Z_o} \hat{\mathbf{k}} \times \mathbf{E} \tag{8.9}$$

If we choose the z-axis to be along $\mathbf{k}$ and the x-axis along $\mathbf{E}_o$, the solution becomes

$$\mathbf{E} = E_o \hat{\mathbf{x}} \cos(kz - \omega t) \tag{8.10}$$

with

$$k^2 = \mu_o \varepsilon_o \omega^2 \tag{8.11}$$

and

$$v = \frac{k}{\omega} = \frac{1}{\sqrt{\varepsilon_o \mu_o}} = c \tag{8.12}$$

In addition to this simple plane wave solution, consider a solution of the form

$$\mathbf{E} = E_o e^{-\alpha y} \hat{\mathbf{x}} \cos(kz - \omega t) \tag{8.13}$$

then

$$\nabla^2 \mathbf{E} = \mu_o \varepsilon_o \omega^2 \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{8.14}$$

gives

$$(\alpha^2 - k^2)\mathbf{E} = -\mu_o \varepsilon_o \omega^2 \mathbf{E} \tag{8.15}$$

or

$$k^2 - \alpha^2 = \mu_o \varepsilon_o \omega^2 \tag{8.16}$$

Now we see that, for a given frequency, a range of k values are allowed (and different wavelengths can occur) depending on α. These solutions, which vary exponentially in the direction perpendicular to $\mathbf{k}$, are called inhomogeneous solutions. Note that, if $\alpha \neq 0$, the wavelength is shorter than in a plane wave ($\alpha = 0$) at the same frequency. These solutions occur in nature, they are usually called 'evanescent waves', and we will encounter them again when we discuss total internal reflection.