(Chapter head:)Polarized Light

1 The Wave Equation as an Eigenvalue Problem

In homogeneous media, the wave equation for the electric field is

$$\nabla \times \nabla \times \mathbf{E} = -\mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{1}$$

and, as $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$,

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{2}$$

If the field has the form

$$\mathbf{E} = \mathbf{E}_o e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{3}$$

this becomes

$$-\mathbf{k}(\mathbf{k} \cdot \mathbf{E}_o) + k^2 \mathbf{E}_o = \mu\varepsilon\omega^2 \mathbf{E}_o \tag{4}$$

and letting $\mathbf{k} = k\hat{\mathbf{k}}$,

$$(\mathbf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}})\mathbf{E}_o = \frac{\mu\omega^2}{k^2} \varepsilon \mathbf{E}_o \tag{5}$$

Usually, the direction of propagation $\hat{\mathbf{k}}$ is known. In anisotropic media ε is a tensor, but in isotropic media it is a scalar, and Eq. 5 may be regarded as a simple eigenvalue equation. The eigenvectors are the allowed values of $\mathbf{E}$, and $\frac{\mu\varepsilon\omega^2}{k^2}$ is the eigenvalue, which determines the velocity of propagation. Since $\omega/k = v = c/n$, and $\mu\varepsilon = \mu_r \varepsilon_r / c^2$, we can write

$$(\mathbf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}})\mathbf{E}_o = \gamma \mathbf{E}_o \tag{6}$$

where $\gamma = (\frac{\mu_r \varepsilon_r}{n^2})$. Without loss of generality we can choose a coordinate system so that $\hat{\mathbf{k}}$ is along the z-axis, then Eq. 6 becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_{ox} \\ E_{oy} \\ E_{oz} \end{bmatrix} = \gamma \begin{bmatrix} E_{ox} \\ E_{oy} \\ E_{oz} \end{bmatrix} \tag{7}$$

The secular equation $\gamma(1-\gamma)^2 = 0$ gives the eigenvalues $\gamma = 1, 1, 0$, and the corresponding normalized eigenvectors

$$\begin{aligned} \gamma &= 1, & \mathbf{E}_o &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \gamma &= 1, & \mathbf{E}_o &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ \gamma &= 0, & \mathbf{E}_o &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned} \tag{8}$$

We see that there is one non-propagating solution, where $\gamma = 0$ ($n = \infty$, and since $v = c/n$, $v = 0$) ; this corresponds to the field $\mathbf{E}$ along $\hat{\mathbf{k}}$. The other two eigenvectors are degenerate, and correspond to solutions with $\gamma = 1$ or $n = \sqrt{\mu_r \varepsilon_r}$. The general solution can then be written as a linear combination of the two propagating eigenmodes,

$$\mathbf{E} = E_{ox} e^{i(kz-\omega t+\phi_1)} \hat{\imath} + E_{oy} e^{i(kz-\omega t+\phi_2)} \hat{\jmath} \tag{9}$$

2 The Polarization Ellipse

We consider the path traced out by the tip of the $\mathbf{E}$ vector in the $x-y$ plane. Writing explicitly the real part,

$$\mathbf{E} = E_{ox} \cos(kz - \omega t - \phi_1)\hat{\imath} + E_{oy} \cos(kz - \omega t - \phi_2)\hat{\jmath} = E_x \hat{\imath} + E_y \hat{\jmath} \tag{10}$$

and writing E_y in terms of E_x by eliminating $\cos(kz - \omega t - \phi_1)$ gives

$$\frac{E_y}{E_{oy}} = \cos\delta \frac{E_x}{E_{ox}} + \sin\delta \sqrt{1 - \left(\frac{E_x}{E_{ox}}\right)^2} \tag{11}$$

where $\delta = \phi_2 - \phi_1$, which may be rearranged to read

$$\left(\frac{E_y}{E_{oy}}\right)^2 - 2\frac{E_y}{E_{oy}}\frac{E_x}{E_{ox}}\cos\delta + \left(\frac{E_x}{E_{ox}}\right)^2 = \sin^2\delta \tag{12}$$

This is the equation of an ellipse, which can be written in the bilinear form

$$\begin{bmatrix} E_x & E_y \end{bmatrix} \begin{bmatrix} \frac{1}{E_{ox}^2} & -\frac{\cos\delta}{E_{ox}E_{oy}} \\ -\frac{\cos\delta}{E_{ox}E_{oy}} & \frac{1}{E_{oy}^2} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \sin^2\delta \tag{13}$$

Recall that if we have the equation of a surface $A_{\alpha\beta} r_a r_b = c$, the eigenvectors give the directions to the extremal points, and the distance to these points from the origin is just.$r = \sqrt{c/\gamma}$ where γ is the eigenvalue. We write

$$\begin{bmatrix} E_x & E_y \end{bmatrix} \begin{bmatrix} 1 & a \\ a & b \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = c \tag{14}$$

where we have factored out $\frac{1}{E_{ox}^2}$;so $a = -\frac{E_{ox}}{E_{oy}} \cos\delta$, $b = \frac{E_{ox}^2}{E_{oy}^2}$ and $c = E_{ox}^2 \sin^2\delta$. We can find the eigenvectors at once. Consider the eigenvalue problem

$$\begin{bmatrix} 1 & a \\ a & b \end{bmatrix} \begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix} = \gamma \begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix} \tag{15}$$

Direct substitution gives

$$\cos\psi + a \sin\psi = \gamma \cos\psi \tag{16}$$

and

$$a\cos\psi + b\sin\psi = \gamma\sin\psi \tag{17}$$

Eliminating the eigenvalue γ gives

$$1 + a\tan\psi = \frac{a}{\tan\psi} + b \tag{18}$$

or

$$1 - b = 2a(\frac{1-\tan^2\psi}{2\tan\psi}) = \frac{2a}{\tan 2\psi} \tag{19}$$

Solving for ψ gives

$$\tan\psi = \frac{E_{ox}^2 - E_{oy}^2}{2E_{oy}^2} \pm \sqrt{(\frac{E_{ox}^2 - E_{oy}^2}{2E_{oy}^2})^2 + \frac{E_{ox}^2}{E_{oy}^2}\cos^2\delta} \tag{20}$$

and

$$\tan 2\psi = \frac{2a}{1-b} = \frac{-2E_{ox}\cos\delta}{E_{oy}(1 - \frac{E_{ox}^2}{E_{oy}^2})} \tag{21}$$

The two solutions (say nearest to $\psi = 0$) give the directions of the axes of the ellipse

$$\tan 2\psi = \frac{2E_{ox}E_{oy}}{E_{ox}^2 - E_{oy}^2}\cos\delta \tag{22}$$

where ψ is the angle measured from the x-axis.

Now we want to determine the lengths of the major and minor axes of the ellipse. Writing $E_x = E_{ext}\cos\psi$ and $E_y = E_{ext}\sin\psi$ where E_{ext} is the extremal value (min or max) of the electric field, Eq.14 gives

$$E_{ext}^2 \begin{bmatrix} \cos\psi & \sin\psi \end{bmatrix} \begin{bmatrix} 1 & a \\ a & b \end{bmatrix} \begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix} = c \tag{23}$$

but since γ is an eigenvalue,

$$E_{ext}^2 = \frac{c}{\gamma} \tag{24}$$

This gives $E_{\max}^2 = c/\gamma_1$, where γ_1 is the smallest eigenvalue, and $E_{\min}^2 = c/\gamma_2$, where γ_2 is the largest eigenvalue.

We get the eigenvalue at once from the secular equation:

$$\begin{vmatrix} 1-\gamma & a \\ a & b-\gamma \end{vmatrix} = (1-\gamma)(b-\gamma) - a^2 = 0 \tag{25}$$

or

$$\gamma = (\frac{1+b}{2}) \pm \sqrt{(\frac{1+b}{2})^2 + a^2 - b} \tag{26}$$

and substitution gives

$$\gamma = \frac{1}{2E_{oy}^2}(E_{ox}^2 + E_{oy}^2) \pm \frac{1}{2E_{oy}^2}\sqrt{E_{ox}^4 + 2\cos 2\delta E_{ox}^2 E_{oy}^2 + E_{oy}^4} \tag{27}$$

The ratio

$$\frac{E_{\min}^2}{E_{\max}^2} = \frac{\gamma_1}{\gamma_2} = \frac{(E_{ox}^2 + E_{oy}^2) - \sqrt{E_{ox}^4 + 2\cos 2\delta E_{ox}^2 E_{oy}^2 + E_{oy}^4}}{(E_{ox}^2 + E_{oy}^2) + \sqrt{E_{ox}^4 + 2\cos 2\delta E_{ox}^2 E_{oy}^2 + E_{oy}^4}} \tag{28}$$

Using the relation for $\tan\psi$, this can also be written as

$$\tan\chi = \frac{E_{\min}}{E_{\max}} = \frac{E_{ox}\sin\phi_1\sin\psi - E_{oy}\sin\phi_2\cos\psi}{E_{ox}\cos\phi_1\cos\psi + E_{oy}\cos\phi_2\sin\psi} \tag{29}$$

The transformation from the original form

to the description of the polarization ellipse is

$$\tan\psi = \frac{E_{ox}^2 - E_{oy}^2}{2E_{oy}^2} \pm \sqrt{(\frac{E_{ox}^2 - E_{oy}^2}{2E_{oy}^2})^2 + \frac{E_{ox}^2}{E_{oy}^2}\cos^2\delta} \tag{30}$$

and

$$\tan^2\chi = \frac{E_{\min}^2}{E_{\max}^2} = \frac{(E_{ox}^2 + E_{oy}^2) - \sqrt{E_{ox}^4 + 2\cos 2\delta E_{ox}^2 E_{oy}^2 + E_{oy}^4}}{(E_{ox}^2 + E_{oy}^2) + \sqrt{E_{ox}^4 + 2\cos 2\delta E_{ox}^2 E_{oy}^2 + E_{oy}^4}} \tag{31}$$

The parameters ψ and χ characterize the polarization of the wave.

2.1 Linear Polarization

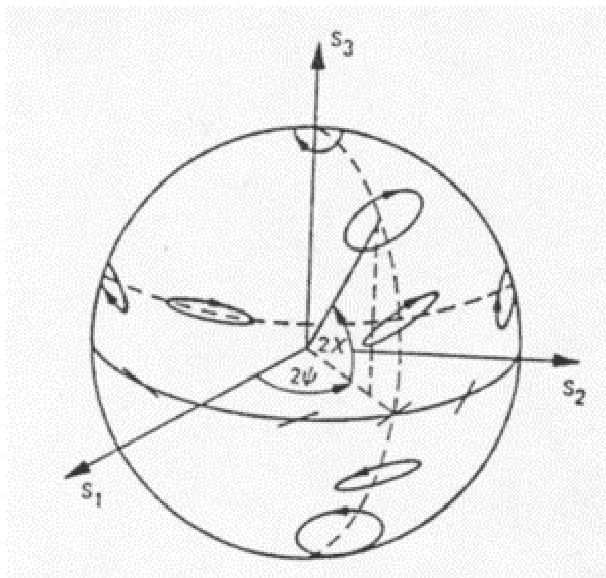
Linear (or plane) polarization is obtained if $\delta = 0$. In this case $\tan\psi = E_{oy}/E_{ox}$ or $\psi = \tan^{-1}(E_{oy}/E_{ox})$ and $\chi = 0$. The electric field oscillates in time, maintaining its direction.

2.2 Circular Polarization

Circular polarization is obtained if $E_{ox} = E_{oy}$ and $\delta = \pm\pi/2$. In this case the electric field vector traces out a circle. In this case ψ is indeterminate, and $\tan\chi = 1$. Due to convention in optics, if the electric field vector rotates *clockwise* in time at a fixed position when viewed with light travelling towards the observer, it is said to be *right circularly polarized*. If it rotates counterclockwise, it is left circularly polarized. (Note that the electric field of right circularly polarized light rotates counterclockwise in space at a fixed time when moving in the direction of propagation).

2.3 Connection with Poincare Sphere

It is useful to associate a polarization state (ψ, χ) with a point on the sphere.
Such a sphere is called the Poincare sphere.
The coordinates of a point on a sphere are shown below:



2.4 The Jones Vector

In general, we can write the expression for a plane wave as

$$\mathbf{E} = E_{ox}\cos(kz - \omega t - \phi_1)\hat{\imath} + E_{oy}\cos(kz - \omega t - \phi_2)\hat{\jmath} = E_x\hat{\imath} + E_y\hat{\jmath} \tag{32}$$

or, in complex notation, as

$$\mathbf{E} = E_{ox}e^{-i\phi_1}e^{i(kz-\omega t)}\hat{\imath} + E_{oy}e^{-i\phi_2}e^{e^{i(kz-\omega t)}} \tag{33}$$

or

$$\mathbf{E} = A_x e^{i(kz-\omega t)}\hat{\imath} + A_y e^{e^{i(kz-\omega t)}} \tag{34}$$

where $A_x = E_{ox}e^{-i\phi_1}$ and $A_y = E_{oy}e^{-i\phi_2}$ are the complex amplitudes of the two components of the field.

It is convenient to write these complex amplitudes in the form of a matrix

$$\mathbf{j} = \begin{bmatrix} A_x \\ A_y \end{bmatrix} \tag{35}$$

This is the Jones vector.

In terms of this, the intensity is given by

$$I = \frac{1}{2Z}(|A_x|^2 + |A_y|^2) \tag{36}$$

and, in order to calculate $\psi$and χ, we have

$$E_{ox} = |A_x| \tag{37}$$

and

$$E_{oy} = |A_y| \tag{38}$$

and

$$\delta = \arg(A_x) - \arg(A_y) \tag{39}$$

Polarization devices, which change the polarization of light (either on transmission or reflection), can be expressed as a matrix. Then

$$\begin{bmatrix} A_{1x} \\ A_{1y} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} A_{2x} \\ A_{2y} \end{bmatrix} \tag{40}$$

or, in compact form,

$$\mathbf{j}_1 = \mathbf{T}\mathbf{j}_2 \tag{41}$$

Examples:

Linear polarizer (in x-direction)

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{42}$$

Wave plate (retarder)

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\Gamma} \end{bmatrix} \tag{43}$$

Polarization rotator

$$\mathbf{T} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{44}$$

Coordinate Transformation Suppose we have a coordinate system where light propagates along the (horizontal) $z-$axis, the y axis is vertical, and the x axis is horizontal.

Suppose now we constuct second coordinate system whose z' axis is parallel to the first, but the $y'-$ (and $x'-$) axes are rotated by the angle θ from the original $y-$ (and $x-$)axes.

If the Jones vector of a polarized light wave in the original coordinate system is $\mathbf{J}$, then, in the second coordinate system, it will be $\mathbf{J}' = \mathbf{R}(\theta)\mathbf{J}$ where

$$\mathbf{R}(\theta)=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{45}$$

If the Jones matrix $\mathbf{T}$ represents an optical system in the original coordinate system, then, in the second coordinate system it will be

$$\mathbf{T}' = \mathbf{R}(\theta)\mathbf{T}\mathbf{R}(-\theta) \tag{46}$$

and conversely,

$$\mathbf{T} = \mathbf{R}(-\theta)\mathbf{T}'\mathbf{R}(\theta) \tag{47}$$

.

This allows us to calculate the Jones matrix of optical elements with arbitrary orientation.

For example, the Jones matrix of the polarizer with easy axis in the $x-$direction

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{48}$$

becomes, if we rotate it by θ,

$$\mathbf{T} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

In a sequence of polarization devices, the total $\mathbf{T}$ matrix is

$$\mathbf{T} = \mathbf{T}_n ... \mathbf{T}_2\mathbf{T}_1 \tag{49}$$

2.5 Field of a Radiating Dipole

To better understand how polarization of light is changed by materials, it us useful to consider radiation by matter.

A fundamental entity is in physics is a dipole. The field of a static dipole is well known, the field of an oscillating dipole is less well known. By solving Maxwell's equations and matching boundary conditions, the electric field of an oscillating dipole of the form

$$\mathbf{P} = P_o\hat{\mathbf{p}}\mathbf{e}^{-i\omega t} \tag{50}$$

in vacuum at the point $\mathbf{r} = r\hat{\mathbf{r}}$ is given by

$$\mathbf{E}_d = \frac{P_o e^{i(kr-\omega t)}}{4\pi\varepsilon_o r^3}[-(kr)^2\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{p}}) - (ikr - 1)(3(\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{I})\} \tag{51}$$

where

$$k = \frac{\omega}{c} \tag{52}$$

The first term goes as $1/r$, the second as $1/r^2$ and the third as $1/r^3$. Far from the dipole, the first term dominates; this gives the radiation field; the resulting intensity goes as $1/r^2$, as required by energy conservation. In the limit that $\omega = 0$, $k = 0$, and the field reduces to the usual static dipole field.

If we ignore the near field terms, the radiated field is

$$\mathbf{E}_d = -\frac{P_o e^{i(kr-\omega t)}}{4\pi\varepsilon_o r} k^2 \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{p}}) = -\pi \frac{P_o e^{i(kr-\omega t)}}{\varepsilon_o r \lambda^2} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{p}}). \tag{53}$$