Chapter 6

Energy Flux

We discuss various ways of describing energy flux and related quantities.

6.0.1 Energy Current Density

The energy current density is given by the Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \tag{6.1}$$

where all quantities are real. The Poynting vector gives the instantaneous local energy current density (energy/(area × time) $[J/(m^2 s) = W/m^2]$).flowing in the direction $\hat{\mathbf{S}}$.

6.0.2 Energy Flux

Flux means current, the amount of flowing 'stuff'/time passing some region of a given (real or imagined) surface. The energy flux is the total radiant energy/time, that is,

$$\Phi = \int \mathbf{S} \cdot d\mathbf{A} = \int (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{A} = \int (\mathbf{E} \times \mathbf{H}) \cdot \hat{\mathbf{N}} dA \tag{6.2}$$

where $\hat{\mathbf{N}}$ is the surface normal. The flux is measured in units of $[W]$.

6.0.3 Irradiance

Irradiance is the energy flux density, that is, the radiant energy/(area×time) incident on a (real or imagined) surface. If the surface normal is $\hat{\mathbf{N}}$, the irradiance is

$$I_i = \mathbf{S} \cdot \hat{\mathbf{N}} = (\mathbf{E} \times \mathbf{H}) \cdot \hat{\mathbf{N}} \tag{6.3}$$

Irradiance is again measured in $[W/m^2]$. Irradiance is sometimes called 'instantenous intensity'.

6.0.4 Radiance

Radiance is the energy flux density per solid angle.$[W/(m^2 \times steradian)]$

6.0.5 Radiant Intensity

Radiant intensity is the energy flux per solid angle $[W/steradian]$
(radiometry)

6.0.6 Intensity

Intensity is the time avaraged irradiance.(time averaged energy flux density).

$$I = \frac{1}{T}\int_0^T \mathbf{S} \cdot \hat{\mathbf{N}} dt = \frac{1}{T}\int_0^T (\mathbf{E} \times \mathbf{H}) \cdot \hat{\mathbf{N}} dt \tag{6.4}$$

Intensity is again measured in $[W/m^2]$

6.0.7 Fluence

Fluence is radiant energy per area of a surface. It is the integrated intensity,

$$Fl = \int_0^T \mathbf{S} \cdot \hat{\mathbf{N}} dt = \int_0^T (\mathbf{E} \times \mathbf{H}) \cdot \hat{\mathbf{N}} dt \tag{6.5}$$

measured in $[J/m^2]$.

6.0.8 Calculating the intensity:

The intensity is given, in general, by

$$I = \frac{1}{T}\int_0^T \mathbf{S}\cdot\hat{\mathbf{N}}dt = \frac{1}{T}\int_0^T (\mathbf{E}\times\mathbf{H})\cdot\hat{\mathbf{N}}dt \tag{6.6}$$

or

$$I = \{\frac{1}{T}\int_0^T \mathbf{S}dt\}\cdot\hat{\mathbf{N}} =< \mathbf{S} > \cdot\hat{\mathbf{N}} \tag{6.7}$$

where we have introduced the time averaged Poynting vector

$$< \mathbf{S} >=\frac{1}{T}\int_0^T \mathbf{S}dt = \frac{1}{T}\int_0^T (\mathbf{E}\times\mathbf{H})dt$$

If the electric and magnetic fields are expressed in complex notation, then

$$< \mathbf{S} > = \frac{1}{4T}\int_0^T (\mathbf{E}+\mathbf{E}^*)\times(\mathbf{H}+\mathbf{H}^*)dt \tag{6.8}$$

$$= \frac{1}{4T}\int_0^T (\mathbf{E}\times\mathbf{H}+\mathbf{E}^*\times\mathbf{H}+\mathbf{E}\times\mathbf{H}^*+\mathbf{E}^*\times\mathbf{H}^*)dt \tag{6.9}$$

If the fields are of the form

$$\mathbf{E} = \mathbf{E}(\mathbf{r})e^{-i\omega t} \tag{6.10}$$

and

$$\mathbf{H} = \mathbf{H}(\mathbf{r})e^{-i\omega t} \tag{6.11}$$

then

$$\int_0^T (\mathbf{E}\times\mathbf{H})dt = \int_0^T (\mathbf{E}^*\times\mathbf{H}^*)dt = 0 \tag{6.12}$$

and

$$\frac{1}{T}\int_0^T (\mathbf{E}^*\times\mathbf{H})dt = \mathbf{E}^*(\mathbf{r})\times\mathbf{H}(\mathbf{r}) = \mathbf{E}^*\times\mathbf{H} \tag{6.13}$$

and

$$\frac{1}{T}\int_0^T (\mathbf{E}\times\mathbf{H}^*)dt = \mathbf{E}(\mathbf{r})\times\mathbf{H}^*(\mathbf{r}) = \mathbf{E}\times\mathbf{H}^* \tag{6.14}$$

and

$$< \mathbf{S} >= \frac{1}{4}(\mathbf{E}^*\times\mathbf{H}+\mathbf{E}\times\mathbf{H}^*) = \frac{1}{2}\,\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*) \tag{6.15}$$

Since

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{6.16}$$

if the magnetic permeability μ is isotropic, it follows that

$$\nabla \times \mathbf{E} = i\mu\omega\mathbf{H} \tag{6.17}$$

and

$$< \mathbf{S} >= \frac{1}{2}\,\mathrm{Re}\,\frac{i}{\mu^*\omega}(\mathbf{E} \times \boldsymbol{\nabla} \times \mathbf{E}^*) \tag{6.18}$$

If the fields are of the form

$$\mathbf{E} = \mathbf{E}_o e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{6.19}$$

and

$$\mathbf{H} = \mathbf{H}_o e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{6.20}$$

it follows that

$$< \mathbf{S} >= \frac{1}{2}\,\mathrm{Re}\,\frac{1}{\mu^*\omega}(\mathbf{E} \times \mathbf{k}^* \times \mathbf{E}^*) \tag{6.21}$$

but also

$$< \mathbf{S} >= \frac{1}{2}\,\mathrm{Re}\,\frac{1}{\mu^*\omega}(\mathbf{E}_o \times \mathbf{k}^* \times \mathbf{E}_o^*)$$

or

$$< \mathbf{S} >= \frac{1}{2}\,\mathrm{Re}\,\frac{1}{\mu^*\omega}((\mathbf{E}_o \cdot \mathbf{E}_o^*)\mathbf{k}^* - \mathbf{E}_o^*(\mathbf{k}^* \cdot \mathbf{E}_o) \tag{6.22}$$

We can always write $\mathbf{k} = k\hat{\mathbf{k}}$. Then

$$< \mathbf{S} >= \frac{1}{2}\,\mathrm{Re}\,\frac{k^*}{\mu^*\omega}((\mathbf{E}_o \cdot \mathbf{E}_o^*)\hat{\mathbf{k}}^* - \mathbf{E}_o^*(\hat{\mathbf{k}}^* \cdot \mathbf{E}_o) \tag{6.23}$$

Now

$$k^2 = \omega^2 \mu\varepsilon \tag{6.24}$$

and so

$$k = \omega\sqrt{\mu\varepsilon} \tag{6.25}$$

so

$$\frac{k^*}{\mu^*\omega} = \frac{\sqrt{\mu^*\varepsilon^*}}{\mu^*} = \sqrt{\frac{\varepsilon^*}{\mu^*}} = \frac{1}{Z^*}$$

and we have

$$< \mathbf{S} >= \frac{1}{2}\,\mathrm{Re}\,\frac{1}{Z^*}((\mathbf{E}_o \cdot \mathbf{E}_o^*)\hat{\mathbf{k}}^* - \mathbf{E}_o^*(\hat{\mathbf{k}}^* \cdot \mathbf{E}_o) \tag{6.26}$$

Now if $\hat{\mathbf{k}}$ is real, then

$$< \mathbf{S} >= \frac{1}{2} \operatorname{Re} \frac{1}{Z^*}((\mathbf{E}_o \cdot \mathbf{E}_o^*)\hat{\mathbf{k}} - \mathbf{E}_o^*(\hat{\mathbf{k}} \cdot \mathbf{E}_o) \tag{6.27}$$

and if ε is isotropic, then

$$\mathbf{k} \cdot \mathbf{E}_o = 0 \tag{6.28}$$

and

$$< \mathbf{S} >= \frac{1}{2} \operatorname{Re} \frac{1}{Z^*}(\mathbf{E}_o \cdot \mathbf{E}_o^*)\hat{\mathbf{k}} = \frac{1}{2}|E_o|^2 \hat{\mathbf{k}} \operatorname{Re}\left(\frac{1}{Z^*}\right) \tag{6.29}$$

The above is the most frequently used form.

The intensity of light, fallling on a plane whose normal is in the $\hat{\mathbf{k}}$ direction, is

$$I = \frac{1}{2} \operatorname{Re}\left(\frac{1}{Z^*}\right)(\mathbf{E}_o \cdot \mathbf{E}_o^*) \tag{6.30}$$