Chapter 5

Plane Waves in Isotropic Media

5.1 Maxwell's Equations:

Maxwell's equations in the absence of free charges in isotropic media are

$$\nabla \cdot \mathbf{D} = 0 \tag{5.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5.3}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \tag{5.4}$$

In addition, we have the constitutive equations

$$\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P} = \varepsilon \mathbf{E} \tag{5.5}$$

and

$$\mathbf{B} = \mu_o (\mathbf{H} + \mathbf{M}) = \mu \mathbf{H} \tag{5.6}$$

If the magnetic permeability is isotropic, the last two of Maxwell's equations can be combined to give the wave equation

$$\nabla \times \nabla \times \mathbf{E} = -\frac{\partial \mu \partial \mathbf{D}}{\partial t^2} \tag{5.7}$$

If μ and ε are independent of time, this can be written as

$$\nabla \times \nabla \times \mathbf{E} = -\mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{5.8}$$

Finally, using the identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{5.9}$$

we get

$$\nabla^2 \mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{5.10}$$

If the material is isotropic,

$$\nabla \cdot \mathbf{D} = \varepsilon \nabla \cdot \mathbf{E} = 0 \tag{5.11}$$

and

$$\nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{5.12}$$

To satisfy Maxwell's equations, $\mathbf{E}$ must satisfy this wave equation, and $\nabla \cdot \mathbf{D} = 0$. Once $\mathbf{E}$ is known, $\mathbf{H}$ can be obtained from $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$.

5.2 Plane Wave Solution:

We try a solution of the form

$$\mathbf{E} = \mathbf{E}_o \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \tag{5.13}$$

This is a plane wave, with amplitude E_o, angular frequency ω, wavelength $\lambda = 2\pi/k$, propagating in the $\hat{\mathbf{k}}$ direction with velocity $v_p = \omega/k$. We call this velocity the wave velocity or phase velocity, hence the subscript p. Substitution into the wave equation Eq. 5.12 gives

$$-k^2 \mathbf{E}_o \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) = -\mu\varepsilon\omega^2 \mathbf{E}_o \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \tag{5.14}$$

and the wave equation is satisfied if

$$k^2 = \omega^2 \mu\varepsilon \tag{5.15}$$

The velocity of propagation is therefore

$$v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} \tag{5.16}$$

We note that

$$\frac{1}{\sqrt{\mu_o \varepsilon_o}} = c = 3 \times 10^8 m/s \tag{5.17}$$

and

$$v_p = \frac{c}{\sqrt{\mu_r \varepsilon_r}} = \frac{c}{n} \tag{5.18}$$

The quantity $n = \sqrt{\mu_r \varepsilon_r}$ is the refractive index, a measure how much slower light propagates in the medium than in vacuum.

Light propagates therefore at different velocities in different media. When light passes from one medium to another, the frequency ω remains constant, but the speed of wave propagation and hence the wavelength change. Since in vacuum

$$c = \frac{\omega}{k_o} \tag{5.19}$$

and in a material

$$\frac{c}{n} = \frac{\omega}{k} \tag{5.20}$$

it is clear that $k = nk_o$, or $\lambda = \lambda_o/n$ As light passes from vacuum into a material, the wavelength becomes shorter.

In addition to the wave equation, Gauss' Law $\nabla \cdot \mathbf{D} = 0$ needs to be satisfied. In isotropic media, this becomes

$$\nabla \cdot \mathbf{E} = 0 \tag{5.21}$$

and substituting the trial solution $\mathbf{E} = \mathbf{E}_o \cos(\mathbf{k} \cdot \mathbf{r} - \omega t)$ gives

$$\mathbf{k} \cdot \mathbf{E}_o \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) = 0 \tag{5.22}$$

or

$$\mathbf{k} \cdot \mathbf{E}_o = 0 \tag{5.23}$$

The field $\mathbf{E}_o$ must therefore be perpendicular to the direction of propagation $\mathbf{k}$. (Note that this only holds for isotropic media.)