Chapter 4

Quantum Mechanics and Classical Fields

Although the classical fields $\mathbf{E}, \mathbf{D}, \mathbf{H}$ and $\mathbf{B}$ give an accurate description of light in many situations, light consists of individual photons, and the appropriate description is quantum mechanical. In this course, we will only briefly touch on the quantum mechanical description.

Maxwell's equations give, for the electric field in vacuum,

$$\nabla^2 \mathbf{E} = \varepsilon_o \mu_o \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{4.1}$$

If the electric field depends only on one spatial coordinate, say, x, then

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} = \varepsilon_o \mu_o \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{4.2}$$

and solutions are

$$\mathbf{E} = \mathbf{E}_o f(x - ct) \tag{4.3}$$

where $\mathbf{E}_o$ is perpendicular to the x-direction.

$$c = \frac{1}{\sqrt{\varepsilon_o \mu_o}} = 3 \times 10^8 m/s \tag{4.4}$$

is the speed of light.

4.1 Energy

Th energy density of the classical radiation field in vacuum is

$$\mathcal{E}/V = \frac{1}{2}\varepsilon_o E^2 + \frac{1}{2}\mu_o H^2 \tag{4.5}$$

Experiments indicate, however, that the energy is not continuous, but is discretized; one may think of the radiation field as consisting of individual energy packets, or photons.

The photoelectric effect, discovered by Hertz, clearly illustrates that each photon carries energy. Monochromatic light consists of photons of a given frequency, and the energy of each photon is

$$\mathcal{E}_{ph} = h\nu \tag{4.6}$$

where $h = 6.6260755 \times 10^{-34} J - s$, and ν is the frequency. In terms of the angular frequency, this is

$$\mathcal{E}_{ph} = \frac{h}{2\pi}\omega = \hbar\omega \tag{4.7}$$

Since photons move at the speed of light, relativistic effects cannot be neglected.

The energy of a particle with mass m is

$$\mathcal{E} = mc^2 = \frac{m_o c^2}{\sqrt{1 - v^2/c^2}} \tag{4.8}$$

where m_o is the rest mass. If photons had (non-zero) rest mass, they would have infinite energy, at variance with experiments. They may be thought of as having zero rest mass but speed c, so in the expression for energy in Eq. 4.8, numerator and denominator are both zero, but the ratio is $h\nu$.

In the expression

$$\mathbf{E} = \mathbf{E}_o f(x - ct) \tag{4.9}$$

for the classical field, f is an arbitrary function - not necessarily a sine or a cosine with a well defined frequency. An arbitrary function f can, however, be expanded in a Fourier series or Fourier transform, in terms of $\cos(kx - \omega t)$ and $\sin(kx - \omega t)$, wher each term has a well defined frequency. One can therefore think of a field with arbitrary functional form as consisting of a collection of photons with different frequencies.

If we have monochromatic light, where the field is given by

$$\mathbf{E} = \mathbf{E}_o \cos(kx - \omega t) \tag{4.10}$$

the energy density is

$$\mathcal{E}/V = \frac{1}{2}\varepsilon_o E^2 + \frac{1}{2}\mu_o H^2 = \varepsilon_o E^2 = \varepsilon_o E_o^2 \cos^2(kx - \omega t) \tag{4.11}$$

Since each photon carries energy $\hbar\omega$, we must have photon number density

$$\rho_{ph} = \frac{\varepsilon_o E^2}{\hbar\omega} = \frac{\varepsilon_o E^2}{h\nu} \tag{4.12}$$

This is just the photon probability density; we may therefore think of the electric (or magnetic) field as corresponding to the wavefunction of a photon.

4.1.1 Photon current

Current (of anything) is defined as the number density times the (average) velocity.

$$\mathbf{J} = \rho\mathbf{v} \tag{4.13}$$

It tells us the flux; that is, how much (of anything) passes through an area per time.

$$\Phi = \int \mathbf{J} d\mathbf{A} \tag{4.14}$$

We therefore have, in general, the photon current

$$\mathbf{J}_{ph} = \rho_{ph}\mathbf{v}_{ph} \tag{4.15}$$

and in our example,

$$\mathbf{J}_{ph} = \rho_{ph} c\hat{\mathbf{x}} \tag{4.16}$$

or

$$J_{ph} = \frac{\varepsilon_o E^2 c}{h\nu} = \frac{\varepsilon_o E^2 \lambda}{h} = \frac{\varepsilon_o E^2}{p_{ph}} \tag{4.17}$$

4.1.2 Energy Current

The energy current is the energy density times the velocity; that is,

$$\mathbf{J}_{en} = \rho_{ph} h\nu c\hat{\mathbf{x}} \tag{4.18}$$

or

$$J_{en} = \varepsilon_o E^2 c \tag{4.19}$$

If we know the energy per area per time of the radiation field, we can calcuate the magnitude of the electric field.

4.2 Momentum

Linear Momentum

The momentum of a particle with mass m is

$$p = mv = \frac{m_o v}{\sqrt{1 - v^2/c^2}} \tag{4.20}$$

One can eliminate the velocity by combining these expressions as follows:

$$\frac{\mathcal{E}^2}{c^2} - p^2 = m_o^2 c^2 \tag{4.21}$$

providing a general relation between energy, momentum and rest mass

$$\mathcal{E} = c^2 \sqrt{m_o^2 + \frac{p^2}{c^2}} \tag{4.22}$$

Since the rest mass of photons is zero, the expression above gives for the photon momentum

$$p_{ph} = \frac{\mathcal{E}}{c} = \frac{h\nu}{c} \tag{4.23}$$

Since the wavelength $\lambda = v/c$, the momentum is

$$p_{ph} = \frac{h}{\lambda} = \hbar k \tag{4.24}$$

Each photon therefore carries linear momentum $p = h/\lambda$.

This connection between momentum and wavelenght was generalized by de Broglie to particles other than photons.

Since photons carry momentum, they exert pressure - radiation pressure - when they are reflected or absorbed.

4.2.1 Angular Momentum

Photons are Bosons, with spin 1; electrons are Fermions, with spin 1/2. Electrons obey the Pauli exclusion principle; photons, being Bosons do not. (There can be more than one photon in the same quantum state.) Having spin 1, each photon carries spin angular momentum

$$\mathbf{L}_s = \pm\hbar \tag{4.25}$$

In addition to spin angular momentum, they also carry 'orbital' angular momentum, which is also quantized. Imagine light shining into a periscope. The radiation pressure light exerts on the mirrors (due to the photons' linear momentum!) gives rise to a torque on the periscope. The beam is displaced as it traverses the periscope; this change in path results in a change in the 'orbital' angular momentum of light.