Chapter 2

Review of Electricity and Magnetism

It is useful to recall the background of Maxwell's Equations. We consider this for each of the four equations separately.

2.1 Gauss' Law:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\int \rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}' \tag{2.1}$$

This is essentially Coulomb's Law; giving the field $\mathbf{E}(\mathbf{r})$ at $\mathbf{r}$ due to charge density $\rho(\mathbf{r}')$ at $\mathbf{r}'$. It follows from this that

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_o} \tag{2.2}$$

In polarizable materials, it is useful to consider induced electric dipoles. In non-polar materials, in the linear regime, the average dipole moment of a molecule is proportional to the applied field $\mathbf{E}$, that is,

$$\mathbf{p} = \varepsilon_o \chi_e^{MOL}\mathbf{E} \tag{2.3}$$

where α is the molecular polarizability. The electric polarization $\mathbf{P} = \mathcal{N}\mathbf{p} = \varepsilon_o \mathcal{N}\chi_e^{MOL}\mathbf{E}$ is the dipole moment per volume, $\mathcal{N}$ is the number density. The electric potential V, defined by $\mathbf{E} = -\boldsymbol{\nabla} V$, due to this polarization is

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\int \mathbf{P}(\mathbf{r}') \cdot \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}' \tag{2.4}$$

which may be written as

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\int_{surface}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{P}(\mathbf{r}')\cdot d\mathbf{\Omega} - \frac{1}{4\pi\varepsilon_o}\int\frac{\nabla'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\cdot d^3\mathbf{r}'$$

We note that $-\nabla\cdot\mathbf{P}(\mathbf{r}')$ is a charge density; it is the density of bound charges ρ_b. Writing $\rho = \rho_b + \rho_f$, where ρ_f is the density of free charges, Eq. 15.2 becomes

$$\nabla\cdot\mathbf{E} = \frac{\rho_f}{\varepsilon_o} - \frac{\nabla\cdot\mathbf{P}}{\varepsilon_o} \tag{2.5}$$

which may be rearranged to give

$$\nabla\cdot(\varepsilon_o\mathbf{E}+\mathbf{P}) = \rho_f \tag{2.6}$$

This leads to the definition of electric displacement $\mathbf{D}$

$$\mathbf{D} = \varepsilon_o\mathbf{E}+\mathbf{P} \tag{2.7}$$

which, since $\mathbf{P} = \mathcal{N}\chi_e^{MOL}\mathbf{E}$, may be written as

$$\mathbf{D} = \varepsilon_o(1+\mathcal{N}\chi_e^{MOL})\mathbf{E} = \varepsilon_o(1+\chi_e)\mathbf{E} = \varepsilon\mathbf{E} \tag{2.8}$$

where $\varepsilon = \varepsilon_o\varepsilon_r$ is the dielectric permittivity. $\varepsilon_o = 8.85\times10^{-12}F/m$ is the dielectric permittivity of free space, ε_r is the dielectric constant, and χ is the dielectric susceptibility.
Gauss' Law is then simply

$$\nabla\cdot\mathbf{D} = \rho_f \tag{2.9}$$

and in the absence of free charges it becomes

$$\nabla\cdot\mathbf{D} = 0 \tag{2.10}$$

Not that the above arguments assumed that the charges were *stationary*, if there are time varying magnetic fields, then the electric field is *not* simply the Coulomb contribution given by Eq. 15.1

2.2 Gauss' Law for the magnetic field:

The Biot-Savart law gives

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_o}{4\pi}\int\mathbf{J}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}' \tag{2.11}$$

where $\mathbf{B}(\mathbf{r})$ is the magnetic field at $\mathbf{r}$ due to the current density $\mathbf{J}(\mathbf{r}')$ at $\mathbf{r}'$. It follows from this that

$$\nabla \cdot \mathbf{B} = 0 \tag{2.12}$$

Note that this also implies that there is no magnetic 'free charge' - i.e. that there are no magnetic monopoles. Taking the curl of both sides of Eq. 15.13 gives the original Ampère's law:

$$\frac{1}{\mu_o}\nabla \times \mathbf{B} = \mathbf{J} \tag{2.13}$$

In magnetizable materials, it is useful to consider the induced magnetization $\mathbf{M}$, the magnetic dipole moment per volume. The magnetic vector potential $\mathbf{A}$, defined by $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, due to this magnetization is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_o}{4\pi}\int \mathbf{M}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}' \tag{2.14}$$

which may be written as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_o}{4\pi}\int_{surface} \frac{\mathbf{M}(\mathbf{r}') \times d\boldsymbol{\Omega}}{|\mathbf{r} - \mathbf{r}'|} + \frac{\mu_o}{4\pi}\int \frac{\nabla \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \tag{2.15}$$

We note that $\nabla \times \mathbf{M}$ is a current density; it is the current density due to bound charges $\mathbf{J}_b$. Writing $\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f$, where $\mathbf{J}_f$ is the current density due to free charges, Eq. 2.13 becomes

$$\frac{1}{\mu_o}\nabla \times \mathbf{B} = \mathbf{J}_f + \nabla \times \mathbf{M} \tag{2.16}$$

which may be rearranged to give

$$\nabla \times (\frac{1}{\mu_o}\mathbf{B} - \mathbf{M}) = \mathbf{J}_f \tag{2.17}$$

This leads to the definition of the magnetic intensity $\mathbf{H}$

$$\mathbf{H} = \frac{1}{\mu_o}\mathbf{B} - \mathbf{M} \tag{2.18}$$

or

$$\nabla \times \mathbf{H} = \mathbf{J}_f \tag{2.19}$$

Thus **H** really plays a role similar to **D** (cf. Eq. 15.12). In materials without permanent magnetization, in the linear regime, the average magnetic moment **m** of a molecule is proportional to the applied field **H**, that is,

$$\mathbf{m} = \chi_m^{MOL} \mathbf{H} \tag{2.20}$$

where χ_m^{MOL} is the molecular magnetic susceptibility. The magnetization $\mathbf{M} = \mathcal{N}\mathbf{m} = \mathcal{N}\chi_m^{MOL}\mathbf{H}$ is the magnetic dipole moment per volume, $\mathcal{N}$ is the number density.Since $\mathbf{M} = \mathcal{N}\chi_m^{MOL}\mathbf{H}$, Eq. 2.18 may be written as

$$\mathbf{B} = \mu_o(1 + \mathcal{N}\chi_m^{MOL})\mathbf{H} = \mu_o(1 + \chi_m)\mathbf{H} = \mu\mathbf{H} \tag{2.21}$$

where $\mu = \mu_o\mu_r$ is the magnetic permeability. $\mu_o = 4\pi \times 10^{-7} H/m$ is the permeability of free space, μ_r is the relative permeability and χ_m is the magnetic susceptibility. Again, note that the above arguments assumed that the currents were *steady*, if there are time varying electric fields, then the magnetic induction is *not* simply the Biot-Savart contribution given by Eq. 15.13.

2.3 Ampère's Law:

Maxwell corrected the original form of Ampère's Law (Eq.2.13) to include displacement current to give

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \tag{2.22}$$

This is essentially the statement that, in the general time-varying case, in addition to the Biot-Savart contribution, time-varying electric fields also act as sources of the magnetic field. It is interesting to note that even with this contribution, $\nabla \cdot \mathbf{B} = 0$.

2.4 Faraday's Law:

Finally, we have Faraday's Law of induction, which, in differential form, is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.23}$$

This is essentially the statement that, in the general time-varying case, in addition to the Coulomb contribution, time-varying magnetic fields act as sources of the electric field. It is interesting to note that even with this contribution, $\nabla \cdot \mathbf{D} = \rho_f$.