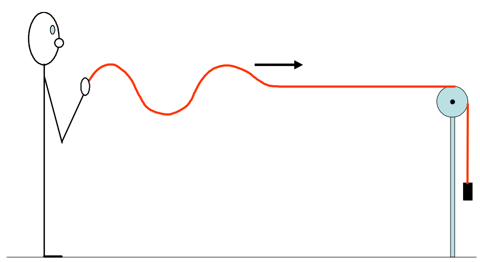
# Chapter 1

# Waves on a String

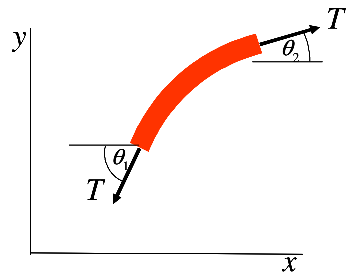
We consider an elastic string, with mass per length $\rho_l$ under constant tension $T$, on which a wave is propagating as shown.



We want to derive the equation of motion of the string, and then consider the properties of the solutions.

## 1.1 The Equations of Motion

We consider a piece of the string, and examine the forces acting on it.




We consider the components of the forces acting on the string at the ends. Since the tension is very nearly constant,

$$F_{1y} = -T\sin\theta_1 \tag{1.1}$$

and

$$F_{2y} = T\sin\theta_2 \tag{1.2}$$

Since $\theta = \theta(x)$, expanding $F_{2y}$ in a Taylor's series gives

$$F_{2y} = T\sin\theta_2 \simeq T\sin\theta_1 + T\cos\theta_1\frac{\partial\theta}{\partial x}dx \tag{1.3}$$

The net force in the $y$-direction is

$$F_y = F_{2y} + F_{1y} = T\cos\theta\frac{\partial\theta}{\partial x}dx \tag{1.4}$$

Now

$$\tan\theta = \frac{\partial y}{\partial x} \tag{1.5}$$

and if $\theta$ is small,

$$\tan\theta \approx \sin\theta \approx \theta = \frac{\partial y}{\partial x} \tag{1.6}$$

It follows that

$$\frac{\partial\theta}{\partial x} = \frac{\partial^2 y}{\partial x^2} \tag{1.7}$$

Since

$$\cos\theta \approx 1 \tag{1.8}$$

we have

$$F_y = T\frac{\partial^2 y}{\partial x^2} \tag{1.9}$$

In the $x$-direction, we have

$$F_{1x} = -T\cos\theta_1 \tag{1.10}$$

and

$$F_{2x} = T\cos\theta_2 = T\cos\theta_1 - T\sin\theta_2\frac{\partial\theta}{\partial x}dx \tag{1.11}$$

and

$$F_x = F_{2x} + F_{1x} = -T\sin\theta_2\frac{\partial\theta}{\partial x}dx \tag{1.12}$$

Since $\theta$ is small, we neglect this term, and assume that, to lowest order, there is no motion in the $x$ direction.

We then have the equation of motion

$$\rho_l dx\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}dx \tag{1.13}$$

and finally we obtain the wave equation

$$\rho_l\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2} \tag{1.14}$$

## 1.2 Solutions

It is interestring to note that if $y$ is of the form

$$y = f(x - vt) \tag{1.15}$$

where $f$ is any continuous twice-differentiable function, then

$$\frac{\partial^2 y}{\partial x^2} = f'' \tag{1.16}$$

where the prime denotes differentiation w.r.t. the argument, and

$$\frac{\partial^2 y}{\partial t^2} = v^2 f'' \tag{1.17}$$

and therefore

$$y = f(x - vt) \tag{1.18}$$

is a solution of the wave equation so long as

$$v = \pm\sqrt{\frac{T}{\rho_l}} \tag{1.19}$$

The quantity $v$ is the wave velocity; it is the speed with which the wave is moving in the $x$ direction.

Since the wave equation is linear, if

$$y = f_1(x - vt) \tag{1.20}$$

is a solution, and

$$y = f_2(x - vt) \tag{1.21}$$

is also a solution, then

$$y = f_1(x - vt) + f_2(x - vt) \tag{1.22}$$

is also a solution.

## 1.3 Energy and Power

### 1.3.1 Energy density

The string has mass and it is moving, hence it has kinetic energy. The kinetic energy of the string, per length, is given by

$$\mathcal{KE}_l = \frac{1}{2}\rho_l(\frac{\partial y}{\partial t})^2 \tag{1.23}$$

Since there is no motion in the $x$-direction, points on the string only move in the $y$-direction. If the slope of a segment changes, the length must also change. Since it takes work to stretch the string, there is also potential energy stored in the string. The change in length of a segment is

$$dl = \sqrt{dx^2 + dy^2} - dx \approx \frac{1}{2}(\frac{\partial y}{\partial x})^2 dx \tag{1.24}$$

Since the force $T$ moves this distance to stretch the string, the potential energy stored, per unit length, is

$$\mathcal{PE}_l = \frac{1}{2}T(\frac{\partial y}{\partial x})^2 \tag{1.25}$$

It is interestring to note that if

$$y = f(x - vt) \tag{1.26}$$

then

$$\mathcal{KE}_l = \frac{1}{2}\rho_l(\frac{\partial y}{\partial t})^2 = \frac{1}{2}\rho_l v^2 f'^2 \tag{1.27}$$

and

$$\mathcal{PE}_l{=}\frac{1}{2}T(\frac{\partial y}{\partial x})^2 = \frac{1}{2}Tf'^2 \tag{1.28}$$

and since

$$v^2 = \frac{T}{\rho_l} \tag{1.29}$$

we find that

$$\mathcal{KE}_l = \frac{1}{2}\rho_l v^2 f'^2 = \frac{1}{2}Tf'^2 = \mathcal{PE}_l \tag{1.30}$$

and kinetic and potential energy densities are the same. Note that this result is only true for a single travelling wave; it is not true if waves travelling in both directions are present!

### 1.3.2 Power

It is interestring to consider the power input to the string by the source (the man waving his arm).

Although he has to exert a force in the $x$-direction to keep the string taut, there is displacement in the $x$-direction, and no associated work.

In the $y$-direction, he has to exert a force

$$F_y = -T\sin\theta \simeq -T\frac{\partial y}{\partial x} \tag{1.31}$$

and the velocity in the $y$-direction is

$$v_y = \frac{\partial y}{\partial t} \tag{1.32}$$

If

$$y = f(x - vt) \tag{1.33}$$

then

$$F_y = -Tf' \tag{1.34}$$

and

$$v_y = -vf' \tag{1.35}$$

and, remarkably, we find that

$$v_y = \frac{v}{T}F_y \tag{1.36}$$

that is, the velocity is proportional to the force!

That is,

$$F_y = \frac{T}{v}v_y = Zv_y \tag{1.37}$$

The quantity

$$Z = \frac{T}{v} = \sqrt{T\rho_l} \tag{1.38}$$

is the impedance.

The input power is

$$\mathcal{P} = F_y v_y = \frac{1}{Z}F_y^2 \tag{1.39}$$

On the string, the total energy density is

$$\mathcal{E}_l = \frac{1}{2}\rho_l(\frac{\partial y}{\partial t})^2 + \frac{1}{2}T(\frac{\partial y}{\partial x})^2 \tag{1.40}$$

and if

$$y = f(x - vt) \tag{1.41}$$

then

$$\mathcal{E}_l = T(\frac{\partial y}{\partial x})^2 = Tf'^2 \tag{1.42}$$

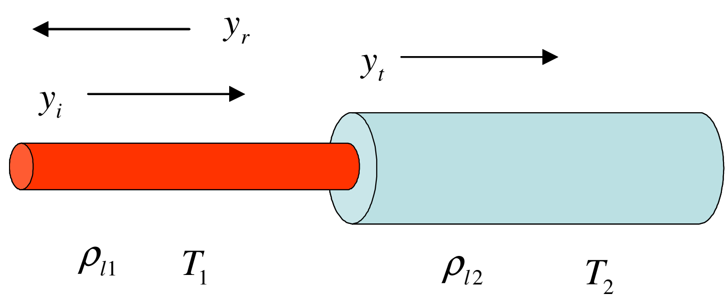
and the total propagating power is

$$\mathcal{P}_p = \mathcal{E}_l v = vTf'^2 = \frac{1}{Z}T^2 f'^2 = \frac{F_y^2}{Z} = \mathcal{P}_{in} \tag{1.43}$$

and the power propagating in the string is equal to the input power, as expected.

## 1.4 Reflection and Transmission

It is interestring to consider two strings which are joined together, as shown.




We want to understand what happens when a wave is incident on the interface. We expect that there will be some reflection, so there will be a reflected wave, and some transmission, so there will be a transmitted wave. There will therefore be three waves: the incident wave

$$y_i = y_i(x - v_1 t) \tag{1.44}$$

a reflected wave

$$y_r = y_r(x + v_1 t) \tag{1.45}$$

and a transmitted wave

$$y_t = y_t(x - v_2 t) \tag{1.46}$$

Let us suppose the interface is at $x = 0$. The boundary conditions at the interface are as follows:

1. The string is continuous, therefore the amplitude $y$ is continuous. This means that

$$y_i(-v_1 t) + y_r(+v_1 t) = y_t(-v_2 t) \tag{1.47}$$

2. The net force on the interface (whose width is zero) must be zero. This means that the total force in the $y$-direction on the interface is zero; that is $T\partial y/\partial x$ is continuous. This gives

$$T_1 \frac{\partial y_i(-v_1 t)}{\partial x} + T_1 \frac{\partial y_r(+v_1 t)}{\partial x} = T_2 \frac{\partial y_t(-v_2 t)}{\partial x} \tag{1.48}$$

Boundary condition 1 suggests that the three functions appearing in

$$y_i(-v_1 t) + y_r(+v_1 t) = y_t(-v_2 t) \tag{1.49}$$

have the same time dependence. We assume therefore

$$y_i(-v_1 t) = g(t) \tag{1.50}$$

and

$$y_r(+v_1 t) = rg(t) \tag{1.51}$$

and

$$y_t(-v_2 t) = tg(t) \tag{1.52}$$

where $r$ and $t$ are constants. In this case, clearly boundary condition 1 is satisfied, so long as

$$1 + r = t \tag{1.53}$$

We can now write, in general,

$$y_i(x - v_1 t) = y_i(-v_1(-\frac{x}{v_1} + t)) = g(-\frac{x}{v_1} + t) \tag{1.54}$$

and

$$y_r(x + v_1 t) = y_r(v_1(\frac{x}{v_1} + t)) = rg(\frac{x}{v_1} + t) \tag{1.55}$$

and

$$y_t(x - v_2 t) = y_t(-v_2(-\frac{x}{v_2} + t)) = tg(-\frac{x}{v_2} + t) \tag{1.56}$$

A key point here is that the time dependence of all three waves is the same! (What is different is the coefficient of the spatial coordinate.) We can now write the reflected and transmitted waves in terms of the incident one:

$$y_r(x + v_1 t) = rg(\frac{x}{v_1} + t) = ry_i(-v_1(+\frac{x}{v_1} + t)) = ry_i(-x - v_1 t) \tag{1.57}$$

where we have just replaced $(-\frac{x}{v_1} + t)$ in the argument of $y_i$ by $(+\frac{x}{v_1} + t)$, and

$$y_t(x - v_2 t) = tg(-\frac{x}{v_2} + t) = ty_i(-v_1(-\frac{x}{v_2} + t)) = ty_i(\frac{v_1}{v_2}(x - v_2 t)) \tag{1.58}$$

Note that the reflected wave is reversed - the argument is $-x - v_1 t$; the spatial variable has changed sign. Also, notice that the transmitted wave is contracted/expanded in the $x-$direction by the ratio of $v_1/v_2$. (We acknowledge Shuang Zhou for pointing this out!)

So if we know $y_i$, we can determine $y_r$ and $y_t$ if the constants $r$ and $t$ are known.

We now consider boundary condition 2. This becomes

$$T_1 \frac{\partial g(-x/v_1+t)}{\partial x}|_{0-} + T_1 r \frac{\partial g(+x/v_1+t)}{\partial x}|_{0-} = T_2 t \frac{\partial g(-x/v_2+t)}{\partial x}|_{0+} \quad (1.59)$$

or

$$(-\frac{T_1}{v_1} + r\frac{T_1}{v_1})g(t) = -t\frac{T_2}{v_2}g(t) \quad (1.60)$$

and

$$Z_1(1-r) = Z_2 t \quad (1.61)$$

or

$$1 - r = \frac{Z_2}{Z_1}t \quad (1.62)$$

Solving this simultaneously with

$$1 + r = t \quad (1.63)$$

gives, for the reflection coefficient $r$,

$$r = \frac{Z_1 - Z_2}{Z_1 + Z_2} \quad (1.64)$$

The magnitude of the reflected wave is therefore determined by the impedance mismatch between the strings. If the impedance is the same (even though the tensions and mass densities may be different) there is no reflection. The transmission coefficient $t$ is

$$t = \frac{2Z_1}{Z_1 + Z_2} \quad (1.65)$$

If a single string is tied to the wall, we can regard the wall as a string of infinite mass density. In this case, $Z_2$ is infinite, and $r = -1$ and $t = 0$.

It is useful to look at power at the interface.

The incident power, from the incident wave, is

$$\mathcal{P}_i = (\frac{1}{2}\rho_l(\frac{\partial y_i}{\partial t})^2 + \frac{1}{2}T_1(\frac{\partial y_i}{\partial x})^2)v_1 = T_1 v_1(\frac{\partial y_i}{\partial x})^2 = \frac{T_1}{v_1}g'^2 = Z_1 g'^2 \quad (1.66)$$

the reflected power is

$$\mathcal{P}_r = T_1 v_1(\frac{\partial y_r}{\partial x})^2 = Z_1 r^2 g'^2 \quad (1.67)$$

and the transmitted power is

$$\mathcal{P}_t = T_2 v_2 (\frac{\partial y_t}{\partial x})^2 = Z_2 t^2 g'^2 \tag{1.68}$$

and since

$$1 - r^2 = \frac{Z_2}{Z_1} t^2 \tag{1.69}$$

we see that

$$\mathcal{P}_i = \mathcal{P}_r + \mathcal{P}_t \tag{1.70}$$

as expected. Energy conservation is therefore implicit in the wave equation and boundary conditions.

Although the material is characterized by two quantities, $\rho_l$ and $T$, it is useful instead to think of it as being characterized by the wave velocity

$$v = \sqrt{\frac{T}{\rho_l}} \tag{1.71}$$

and the impedance

$$Z = \sqrt{T \rho_l} \tag{1.72}$$

The efficiency of energy transfer to and from the material depends on the impedance match between the source and the material.

It is useful to think of the tension $T$ in terms of the strain $e = \Delta l / l$ and Young's modulus; if the cross-sectional area of the string is $A$, then

$$T = eYA \tag{1.73}$$

and the mass per length $\rho_l$ can be written in terms of the density $\rho$ as

$$\rho_l = \rho A \tag{1.74}$$

In terms of bulk properties, then, we get for the wave velocity

$$v = \sqrt{e \frac{Y}{\rho}} \tag{1.75}$$

and for the impedance

$$Z = A \sqrt{eY\rho} \tag{1.76}$$

The units of impedance are momentum density $\times$ area.

### 1.4.1 Counterpropagating Waves

Consider two waves, propagating in opposite direction. Then

$$y = f_1(x - vt) + f_2(-x - vt) \tag{1.77}$$

The kinetic energy per length is

$$\mathcal{KE}_l = \frac{1}{2}\rho(\frac{\partial y}{\partial t})^2 = \frac{1}{2}\rho(-vf_1' - vf_2')^2 = \frac{1}{2}\rho v^2(f_1'^2 + 2f_1'f_2' + f_2'^2) \tag{1.78}$$

the potential energy per length is

$$\mathcal{PE}_l = \frac{1}{2}T(\frac{\partial y}{\partial x})^2 = \frac{1}{2}T(f_1' - f_2')^2 = \frac{1}{2}T(f_1'^2 - 2f_1'f_2' + f_2'^2) \tag{1.79}$$

and nothing the $\rho v^2 = T$, the total energy is

$$\mathcal{E}_l = Tf_1'^2 + Tf_2'^2 = \mathcal{E}_{l1} + \mathcal{E}_{l2} \tag{1.80}$$

Due to the cancellation of the cross terms, the total energy of the sum of the two counterpropagating waves is just the sum of the energies of the individual waves. This is NOT the case if the waves are travelling in the same direction..

## 1.5 Summary

Transverse waves on a string obey the wave equation

$$T\frac{\partial^2 y}{\partial x^2} = \rho\frac{\partial^2 y}{\partial t^2} \tag{1.81}$$

Solutions are functions of the form

$$y = f(x \pm vt) \tag{1.82}$$

where the wave velocity is

$$v = \sqrt{\frac{T}{\rho}} \tag{1.83}$$

and the impedance is

$$Z = \sqrt{T\rho}$$

A useful form is

$$y = y_o \cos(kx - \omega t) \tag{1.84}$$

where

$$k = \frac{2\pi}{\lambda} \tag{1.85}$$

is the wavenumber and

$$\omega = \frac{2\pi}{T} = 2\pi f \tag{1.86}$$

is the angular frequency. Clearly

$$v = \frac{\omega}{k} = \lambda f \tag{1.87}$$

They carry kinetic and potential energy; these are, per length,

$$\mathcal{KE}_l = \frac{1}{2}\rho_l(\frac{\partial y}{\partial t})^2 \tag{1.88}$$

and

$$\mathcal{PE}_l = \frac{1}{2}T(\frac{\partial y}{\partial x})^2 \tag{1.89}$$

If a wave $y = y_i(x - v_1 t)$ is incident on an interface, there will be a reflected wave

$$y_r = r y_i(-x - v_1 t) \tag{1.90}$$

and a transmitted wave

$$y_t = t y_i(\frac{v_1}{v_2}(x - v_2 t)) \tag{1.91}$$

where the reflection coefficient is

$$r = \frac{Z_1 - Z_2}{Z_1 + Z_2} \tag{1.92}$$

and the transmission coefficient is

$$t = \frac{2Z_2}{Z_1 + Z_2} \tag{1.93}$$

The reflected plus the transmitted power is equal to the incident power; that is, energy is conserved by solutions of the wave equation with the appropriate boundary conditions.