(Chapter head:)Interference

We consider two plane waves incident on a plane surface. In real notation,

$$\mathbf{E}_1 = \mathbf{E}_{o1} \cos(\mathbf{k}_1 \cdot \mathbf{r}_1 - \omega_1 t + \phi_1) \tag{1}$$

and

$$\mathbf{E}_2 = \mathbf{E}_{o2} \cos(\mathbf{k}_2 \cdot \mathbf{r}_2 - \omega_2 t + \phi_2) \tag{2}$$

The Poynting vector is $\mathbf{S} = \mathbf{E} \times \mathbf{H}$. If the normal to the surface on which the waves are incident is $\hat{\mathbf{n}}$, then the irradiance (instanteneous intensity) is

$$I_i = \mathbf{S} \cdot \hat{\mathbf{n}} = (\mathbf{E} \times \mathbf{H}) \cdot \hat{\mathbf{n}} \tag{3}$$

and writing $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ gives

$$\begin{aligned} I_i &= ((\mathbf{E}_1 + \mathbf{E}_2) \times (\mathbf{H}_1 + \mathbf{H}_2)) \cdot \hat{\mathbf{n}} \\ &= (\mathbf{E}_1 \times \mathbf{H}_1 + \mathbf{E}_1 \times \mathbf{H}_2 + \mathbf{E}_2 \times \mathbf{H}_1 + \mathbf{E}_2 \times \mathbf{H}_2) \cdot \hat{\mathbf{n}} \end{aligned} \tag{4}$$

since $\mathbf{H} = \hat{\mathbf{k}} \times \mathbf{E}/Z$ where Z is the impedance, this becomes

$$I_i = \frac{1}{Z}(\mathbf{E}_1 \times \hat{\mathbf{k}}_1 \times \mathbf{E}_1 + \mathbf{E}_1 \times \hat{\mathbf{k}}_2 \times \mathbf{E}_2 + \mathbf{E}_2 \times \hat{\mathbf{k}}_1 \times \mathbf{E}_1 + \mathbf{E}_2 \times \hat{\mathbf{k}}_2 \times \mathbf{E}_2) \cdot \hat{\mathbf{n}} \tag{5}$$

and noting that $\mathbf{A} \times \mathbf{B} \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

$$\begin{aligned} I_i &= \frac{1}{Z}(\hat{\mathbf{k}}_1 E_1^2 + \hat{\mathbf{k}}_2 E_2^2 + \\ &\quad (\hat{\mathbf{k}}_1 + \hat{\mathbf{k}}_2)(\mathbf{E}_1 \cdot \mathbf{E}_2) - \mathbf{E}_2(\mathbf{E}_1 \cdot \hat{\mathbf{k}}_2) - \mathbf{E}_1(\mathbf{E}_2 \cdot \hat{\mathbf{k}}_1)) \cdot \hat{\mathbf{n}} \end{aligned} \tag{6}$$

and we have made use of $\mathbf{E}_i \cdot \hat{\mathbf{k}}_i = 0$. Now we write the time and position dependence explicitly

$$\begin{aligned} I_i &= \frac{1}{Z}((\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{n}}) E_{o1}^2 \cos^2(\mathbf{k}_1 \cdot \mathbf{r}_1 - \omega_1 t + \phi_1) + \\ &\quad (\hat{\mathbf{k}}_2 \cdot \hat{\mathbf{n}}) E_{o2}^2 \cos^2(\mathbf{k}_2 \cdot \mathbf{r}_2 - \omega_2 t + \phi_2) + \\ &\quad \hat{\mathbf{n}} \cdot ((\hat{\mathbf{k}}_1 + \hat{\mathbf{k}}_2)(\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_2) - \hat{\mathbf{E}}_1(\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{k}}_1) - \hat{\mathbf{E}}_2(\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{k}}_2)) \times \\ &\quad E_{o1} E_{o2} \cos(\mathbf{k}_1 \cdot \mathbf{r}_1 - \omega_1 t + \phi_1) \cos(\mathbf{k}_2 \cdot \mathbf{r}_2 - \omega_2 t + \phi_2) \end{aligned} \tag{7}$$

We note that the first two terms on the r.h.s. are intensities of the individual waves (if the other wave was not present). and the last is the interference term. Taking the time average gives the intensity

$$\begin{aligned} I &= \frac{1}{Z}(\frac{1}{2}(\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{n}}) E_{o1}^2 + \frac{1}{2}(\hat{\mathbf{k}}_2 \cdot \hat{\mathbf{n}}) E_{o2}^2 + \\ &\quad \frac{1}{2}\alpha E_{o1} E_{o2} < [\cos(\mathbf{k}_1 \cdot \mathbf{r}_1 + \mathbf{k}_2 \cdot \mathbf{r}_2 - (\omega_1 + \omega_2)t + \phi_1 + \phi_2) + \\ &\quad \cos(\mathbf{k}_1 \cdot \mathbf{r}_1 - \mathbf{k}_2 \cdot \mathbf{r}_2 - (\omega_1 - \omega_2)t + \phi_1 - \phi_2)] >) \end{aligned} \tag{8}$$

where $\alpha = \hat{\mathbf{n}}\cdot((\hat{\mathbf{k}}_1 + \hat{\mathbf{k}}_2)(\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_2) - \hat{\mathbf{E}}_1(\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{k}}_1) - \hat{\mathbf{E}}_2(\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{k}}_2))$ and $<>$ denotes the time average. Now $< \cos(\mathbf{k}_1 \cdot \mathbf{r}_1 + \mathbf{k}_2 \cdot \mathbf{r}_2 - (\omega_1 + \omega_2)t + \phi_1 + \phi_2) >= 0$, and so

$$\begin{aligned} I \quad = \quad & \frac{1}{Z}(\frac{1}{2}(\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{n}})E_{o1}^2 + \frac{1}{2}(\hat{\mathbf{k}}_2 \cdot \hat{\mathbf{n}})E_{o2}^2 + \\ & \frac{1}{2}\alpha E_{o1}E_{o2} < \cos(\mathbf{k}_1 \cdot \mathbf{r}_1 - \mathbf{k}_2 \cdot \mathbf{r}_2 - (\omega_1 - \omega_2)t + \phi_1 - \phi_2)) >) \end{aligned} \tag{9}$$

We note that the interference term vanishes unless $\omega_1 = \omega_2$; that is, *interference only occurs if the light waves have the same frequency.* If this is the case, we write

$$I = I_1 + I_2 + 2\beta\sqrt{I_1 I_2}\cos\delta \tag{10}$$

where β is a geometric factor,

$$\beta = \frac{\hat{\mathbf{n}}\cdot((\hat{\mathbf{k}}_1 + \hat{\mathbf{k}}_2)(\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_2) - \hat{\mathbf{E}}_1(\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{k}}_1) - \hat{\mathbf{E}}_2(\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{k}}_2))}{2\sqrt{(\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{n}})}\sqrt{(\hat{\mathbf{k}}_2 \cdot \hat{\mathbf{n}})}} \tag{11}$$

and δ is the phase difference between the two waves;

$$\delta = -\mathbf{k}_1 \cdot \mathbf{r}_1 + \mathbf{k}_2 \cdot \mathbf{r}_2 + \phi_1 - \phi_2 \tag{12}$$

Finally, we note that $\cos\delta = 2\cos^2\delta/2 - 1$, and

$$I = I_1 + I_2 - 2\beta\sqrt{I_1 I_2} + 4\beta\sqrt{I_1 I_2}\cos^2\delta/2 \tag{13}$$

This is our main result: the interference contribution to the intensity is ˜ $\cos^2\delta/2$ where δ is the phase difference between the waves. In the special case where the polarizations are the same, that is, $\hat{\mathbf{E}}_1 = \hat{\mathbf{E}}_2$ and incidence is such that $\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{n}} = \hat{\mathbf{k}}_2 \cdot \hat{\mathbf{n}}$, then $\beta = 1$, and the above reduces to

$$I = I = (\sqrt{I_1} - \sqrt{I_2})^2 + 4\sqrt{I_1 I_2}\cos^2\delta/2 \tag{14}$$

and if the intensities are the same so that $I_1 = I_2 = I_o$ then

$$I = 4I_o\cos^2\delta/2 \tag{15}$$