Chapter 10

Total Internal Reflection

We have shown that the reflection coefficient for σ polarized light is

$$r_\sigma = \frac{Z_2 \cos\theta_i - Z_1 \cos\theta_t}{Z_2 \cos\theta_i + Z_1 \cos\theta_t} \tag{10.1}$$

and the transmission coefficient is

$$t_\sigma = \frac{2Z_2 \cos\theta_i}{Z_2 \cos\theta_i + Z_1 \cos\theta_t} \tag{10.2}$$

If the material is non-magnetic, then these become

$$r_\sigma = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t} \tag{10.3}$$

and

$$r_\sigma = \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t} \tag{10.4}$$

Similiarly, we have shown that the reflection coefficient for π polarized light is

$$r_\pi = \frac{Z_1 \cos\theta_i - Z_2 \cos\theta_t}{Z_1 \cos\theta_i + Z_2 \cos\theta_t} \tag{10.5}$$

and

$$t_\pi = \frac{2Z_2 \cos\theta_i}{Z_1 \cos\theta_i + Z_2 \cos\theta_t} \tag{10.6}$$

If the material is non-magnetic, then these become

$$r_\pi = \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t} \tag{10.7}$$

and

$$t_\pi = \frac{2n_1 \cos\theta_i}{n_2\cos\theta_i + n_1\cos\theta_t} \tag{10.8}$$

To obtain θ_t, we need to use Snell's Law:

$$n_1 \sin\theta_i = n_2 \sin\theta_t \tag{10.9}$$

This is just the statement that the projection of the incident wave vector and the transmitted wave vector onto the interface are equal. Clearly we can write

$$\mathbf{k}_t = k_t \sin\theta_t \hat{\mathbf{z}} - k_t \cos\theta_t \hat{\imath} \tag{10.10}$$

where $\hat{\mathbf{z}}$ is the direction of $\mathbf{k}_i$ in the plane of the interface, and $\hat{\imath}$ is the normal (pointing from 2 to 1).

From Maxswell's equaitons we get

$$k_t^2 = \omega^2 \mu_2 \varepsilon_2 = \frac{\omega^2}{c^2} n_2 \tag{10.11}$$

and so

$$\mathbf{k}_t = \frac{\omega}{c} n_2 (\sin\theta_t \hat{\mathbf{z}} - \cos\theta_t \hat{\imath})$$

Now suppose that $n_1 > n_2$. There will exist a critical angle of incidence θ_{ic}, such that the corresponding $\theta_t = \pi/2$; here

$$n_1 \sin\theta_{ic} = n_2 \tag{10.12}$$

or

$$\sin\theta_{ic} = \frac{n_2}{n_1} \tag{10.13}$$

At this point, $\sin\theta_t = 1$, and $\cos\theta_t = 0$. What happens if $\theta_i > \theta_{ic}$?

We can always write

$$e^{i\theta_t} = \cos\theta_t + i\sin\theta_t \tag{10.14}$$

and it follows that

$$i\theta_t = \ln(i\sin\theta_t + \sqrt{1-\sin^2\theta_t}) \tag{10.15}$$

If $\sin^2\theta > 0$, it is useful to write this as

$$i\theta_t = \ln(i\sin\theta_t + i\sqrt{\sin^2\theta_t - 1}) = \ln i + \ln(\sin\theta_t + \sqrt{\sin^2\theta_t - 1}) \tag{10.16}$$

Since $\ln i = i\frac{\pi}{2}$,

$$\theta_t = \frac{\pi}{2} - i\ln(\sin\theta_t + \sqrt{\sin^2\theta_t - 1}) \tag{10.17}$$

and

$$\cos\theta_t = i\sqrt{\sin^2\theta_t - 1} \tag{10.18}$$

This gives for $\mathbf{k}_t$

$$\mathbf{k}_t = k_t \sin\theta_t \hat{\mathbf{z}} - k_t \cos\theta_t \hat{\imath} = \frac{\omega}{c} n_2 (\sin\theta_t \hat{\mathbf{z}} - i\sqrt{\sin^2\theta_t - 1}\hat{\imath}) \tag{10.19}$$

and writing $\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i$, we get

$$\mathbf{k}_t = \frac{\omega}{c}(n_1 \sin\theta_i \hat{\mathbf{z}} - i\sqrt{n_1^2 \sin^2\theta_i - n_2^2}\hat{\imath}) \tag{10.20}$$

as before.

So in this regime, Snell's law holds (as always), $\sin\theta_t > 1$, θ_t is complex, $\cos\theta_t$ is imaginary, and $\mathbf{k}_t = k_t \sin\theta_t \hat{\mathbf{z}} - k_t \cos\theta_t \hat{\imath}$ (as always). This corresponds to the usual inhomogeneous solution.

Since $\cos\theta_t$ is imaginary,

$$|r_\sigma| = |r_\pi| = 1 \tag{10.21}$$

and all of the incient energy is reflected. The angle of reflection is the same as the angle of incidence (as always).