Optics I: Theory CPHY 6/74495
Assignment 6 Solutions.
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1. A nematic liquid crystal cell, consisting of two parallel glass plates separated by a distance of $d = 25\mu m$, is oriented so that the plates are in the $x - y$ plane, (the normal to the plates is in the z direction).
   Plane polarized light, polarized along the $\hat{\mathbf{x}}$ direction, is normally incident on the cell.
   The nematic director $\hat{\mathbf{n}}$ is in the $(1, 1, 1,)$ direction everywhere inside the cell.
   The refractive indices of the liquid crystal are $n_e = 1.7$ and $n_o = 1.4$.
   A polarizer in the $x - y$ plane is placed after the cell.. It can be rotated about the z axis, its orientation is defined by the angle θ, such that when $\theta = 0$, the polarizer transmits light polarized along the $\hat{\mathbf{x}}$, axis.

   (a) Plot the normalized intensity of light transmitted by the polarizer as function of the angle θ.

   This is a non-trivial problem, so I give all the details below

   The director is
   $$\hat{\mathbf{n}} = \frac{1}{\sqrt{3}}(1, 1, 1) \tag{1}$$
   The propagating normal modes inside are
   the extraordinary mode:
   $$\mathbf{D}_e = D_e \hat{\mathbf{D}}_e e^{i(\frac{2\pi n}{\lambda_o} z - \omega t)} \tag{2}$$
   and $\hat{\mathbf{D}}_e$ is in the plane normal to $\mathbf{k}$ (the x-y plane) in the direction of the projection of $\hat{\mathbf{n}}$ in this plane, that is,
   $$\hat{\mathbf{D}}_e = \frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath} \tag{3}$$

The electric field corresponding to $\mathbf{D}_e$ is

$$\mathbf{E}_e^{tot} = \varepsilon^{-1}\mathbf{D}_e = \frac{1}{\varepsilon_o}(\frac{1}{\varepsilon_\perp}\mathbf{I} + (\frac{1}{\varepsilon_{||}} - \frac{1}{\varepsilon_\perp})\hat{\mathbf{n}}\hat{\mathbf{n}})\mathbf{D}_e \tag{4}$$

and since

$$\hat{\mathbf{n}} \cdot \hat{\mathbf{D}} = \sqrt{\frac{2}{3}} \tag{5}$$

we get

$$\mathbf{E}_e^{tot} = \frac{1}{\varepsilon_o}(\frac{1}{\varepsilon_\perp}(\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}) + \sqrt{\frac{2}{3}}(\frac{1}{\varepsilon_{||}} - \frac{1}{\varepsilon_\perp})\hat{\mathbf{n}})D_e e^{i(\frac{2\pi n}{\lambda_o}z - \omega t)} \tag{6}$$

The component of this in the x-y plane is

$$\mathbf{E}_e = \frac{1}{\varepsilon_o}(\frac{1}{\varepsilon_\perp}\frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath}) + \frac{\sqrt{2}}{3}(\frac{1}{\varepsilon_{||}} - \frac{1}{\varepsilon_\perp})(\hat{\imath} + \hat{\jmath}))D_e e^{i(\frac{2\pi n}{\lambda_o}z - \omega t)} \tag{7}$$

or simply

$$\mathbf{E}_e = E_e(\hat{\imath} + \hat{\jmath})e^{i(\frac{2\pi n}{\lambda_o}z - \omega t)} \tag{8}$$

where

$$n = \frac{n_e n_o}{\sqrt{n_o^2 + (n_e^2 - n_o^2)(\hat{\mathbf{n}} \cdot \hat{\mathbf{k}})^2}} = \frac{n_e n_o}{\sqrt{n_o^2 + (n_e^2 - n_o^2)\frac{1}{3}}} \tag{9}$$

The ordinary mode is

$$\mathbf{D}_o = D_o\hat{\mathbf{D}}_o e^{i(\frac{2\pi n_o}{\lambda_o}z - \omega t)} = D_o(\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath})e^{i(\frac{2\pi n_o}{\lambda_o}z - \omega t)} \tag{10}$$

and the electric field is

$$\mathbf{E}_o = E_o(\hat{\imath} - \hat{\jmath})e^{i(\frac{2\pi n_o}{\lambda_o}z - \omega t)} \tag{11}$$

and at the exit surface we have

$$\mathbf{E}_e^{out} = E_e(\hat{\imath} + \hat{\jmath})e^{i(\frac{2\pi n}{\lambda_o}d - \omega t)} \tag{12}$$

and

$$\mathbf{E}_o^{out} = E_o(\hat{\imath} - \hat{\jmath})e^{i(\frac{2\pi n_o}{\lambda_o}d - \omega t)} \tag{13}$$

(Note that this corresponds to incident light polarized along the x-axis.)
The polarizer has an easy axis given by

$$\hat{\mathbf{N}} = (\cos\theta, \sin\theta, 0) \tag{14}$$

and the fields after the polarizer are

$$\mathbf{E}_e^{pout} = \hat{\mathbf{N}} \cdot E_e(\hat{\imath} + \hat{\jmath})e^{i(\frac{2\pi n}{\lambda_o}d - \omega t)} = E_e(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})e^{i(\frac{2\pi n}{\lambda_o}d - \omega t)} \tag{15}$$

and

$$\mathbf{E}_o^{pout} = \hat{\mathbf{N}}\cdot E_o(\hat{\imath}-\hat{\jmath})e^{i(\frac{2\pi n_o}{\lambda_o}d-\omega t)} = E_o(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath})e^{i(\frac{2\pi n_o}{\lambda_o}d-\omega t)} \tag{16}$$

Taking the real parts gives

$$\mathbf{E}_e^{pout} = E_e(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})\cos(\frac{2\pi n}{\lambda_o}d-\omega t) \tag{17}$$

and

$$\mathbf{E}_o^{pout} = E_o(\cos\theta\hat{\imath} - \sin\boldsymbol{\theta}\hat{\jmath})\cos(\frac{2\pi n_o}{\lambda_o}d-\omega t) \tag{18}$$

and the intensity is

$$I = \frac{1}{Z_o} < (\mathbf{E}_e^{out} + \mathbf{E}_o^{out})^2 >_t \tag{19}$$

or

$$I = \frac{1}{Z_o}(\frac{1}{2}E_e^2 \tag{20}$$

$$+2E_eE_o(\cos^2\theta - \sin^2\theta) < \cos(\frac{2\pi n}{\lambda_o}d-\omega t)\cos(\frac{2\pi n_o}{\lambda_o}d-\omega t) > +\frac{1}{2}E_o^2) \tag{21}$$

and noting that

$$< \cos(\frac{2\pi n}{\lambda_o}d-\omega t)\cos(\frac{2\pi n_o}{\lambda_o}d-\omega t) >= \frac{1}{2}\cos(\frac{2\pi(n-n_o)d}{\lambda_o}) \tag{22}$$

and finally

$$I = \frac{1}{Z_o}(\frac{1}{2}E_e^2 + E_eE_o(\cos^2\theta - \sin^2\theta)\cos(\frac{2\pi(n-n_o)d}{\lambda_o}) + \frac{1}{2}E_o^2) \tag{23}$$

or

$$I = \frac{1}{2Z_o}(E_e^2 + 2E_eE_o\cos 2\theta\cos(\frac{2\pi(n-n_o)d}{\lambda_o}) + E_o^2) \tag{24}$$

The incident light is polarized in the x-direction, that is

$$\mathbf{E}_{in} = E_{in}\hat{\imath} \tag{25}$$

which can be written in terms of the normal mode amplitudes as

$$\mathbf{E}_{in} = E_{in}\hat{\imath} = E_e(\hat{\imath}+\hat{\jmath}) + E_o(\hat{\imath}-\hat{\jmath}) \tag{26}$$

and we see that $E_e = E_o = \dot{E}_{in}/2$, and

$$I = \frac{1}{8Z_o}(E_{in}^2 + 2E_{in}^2\cos 2\theta\cos(\frac{2\pi(n-n_o)d}{\lambda_o}) + E_{in}^2) \tag{27}$$

$$= \frac{E_{in}^2}{4Z_o}(1 + \cos 2\theta\cos(\frac{2\pi(n-n_o)d}{\lambda_o})) \tag{28}$$

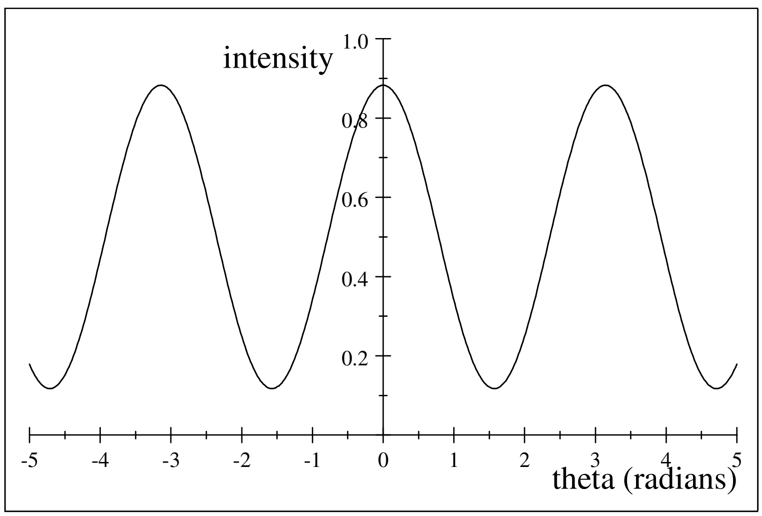
or

$$I = \frac{1}{2} I_o (1 + \cos 2\theta \cos(\frac{2\pi(n - n_o)d}{\lambda_o})) \tag{29}$$

where I_o is the intensity of the incident light. Note that the amplitude of the intensity oscillations as the the polarizer is turned depends on the optical path difference between the two waves. On the average (averaging over polarizer directions, the intensity is $E_{in}^2/4Z_o$, one half of the incident intensity, since the polarizer absorbs half. Evaluating numerically, we get , since $n_o = 1.4$ and $n_e = 1.7$

$$n = \frac{n_e n_o}{\sqrt{n_o^2 + (n_e^2 - n_o^2)\frac{1}{3}}} = 1.580 \tag{30}$$

Now we plot the normalized intensity as function of θ if $\lambda_o = 0.6328$;



The form of this function is the same for all wavelengths; the amplitude at $\theta = 0$ is given by

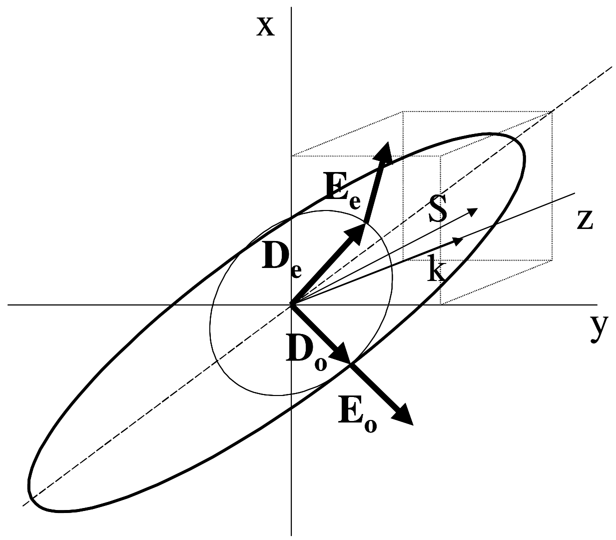
$$\frac{I}{I_o} = \frac{1}{2}(1 + \cos(\frac{2\pi(n - n_o)d}{\lambda_o})) \tag{31}$$

(b) Calculate the angle between the wave vector and the Poynting vectors inside the cell.

$$\tan\beta = \frac{(n_o^2 - n_e^2)(\frac{1}{\sqrt{3}})\sqrt{1 - \frac{1}{3}}}{n_o^2 + (n_e^2 - n_o^2)\frac{1}{3}} = 0.193\,13 \tag{32}$$

and $\beta = 10.93\,\text{deg}$

(c) Sketch the index ellipsoid, and show the fields and the wave and Poynting vectors for both ordinary and extraordinary modes.



For the ordinary wave alone, the Poynting vector is along $\mathbf{k}$. For the extraordinary wave, the Poynting vector is in the plane defined by $\mathbf{k}$ and $\hat{\mathbf{n}}$. If we look at the Poynting vector for the sum, we get

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = (\mathbf{E}_e + \mathbf{E}_o) \times (\mathbf{H}_e \times \mathbf{H}_o) \tag{33}$$

which gives

$$\mathbf{S} = \mathbf{E}_e \times \mathbf{H}_e + \mathbf{E}_e \times \mathbf{H}_o + \mathbf{E}_o \times \mathbf{H}_e + \mathbf{E}_o \times \mathbf{H}_o \tag{34}$$

Now $\mathbf{E}_o \times \mathbf{H}_e = 0$, since $\mathbf{E}_o$ and $\mathbf{H}_e$ are in the same direction. $\mathbf{E}_e \times \mathbf{H}_o$ is not equal to zero in general, however, since the extraordinary and ordinary waves travel at different speeds, the phase difference between them varies, and the spatial average of this is zero. So the spatially averaged Poynting vector is just the sum of the Poynting vector of the extraordinary and the ordinary waves.