Optics I: Theory CPHY 6/72250
Assignment 3.

P. Palffy-Muhoray

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Due: Oct. 13, 2016

1. (19)One end of a very long elastic string is attached to a wall, a person is holding the other end. The mass density of the string is $\rho = 0.1kg/m$ and the tension in the string is $T = 0.9N$. The person is moving the end of string up and down so as to generate a wave, with frequency $f = 3Hz$ and amplitude $y_o = 0.5m$, which travels towards the wall. (Ignore reflections.)

   (a) (2) What is the impedance of the string?

   The impedance is

$$Z = \sqrt{T\rho} = \sqrt{0.9N \times 0.1kg/m} = 0.3kg/s \quad (1)$$

   (b) (4) Give an expression for the energy density in the string.

   The kinetic energy density is

$$E_K = \frac{1}{2}\rho(\frac{\partial y}{\partial t})^2 \quad (2)$$

   The potential energy density is

$$E_P = \frac{1}{2}T(\frac{\partial y}{\partial x})^2 \quad (3)$$

   The expression for the wave is

$$y = y_o \cos(kx - \omega t) \quad (4)$$

   so that

$$E_K = \frac{1}{2}\rho(\frac{\partial y}{\partial t})^2 = \frac{1}{2}\rho\omega^2 y_o^2 \sin^2(kx - \omega t) \quad (5)$$

   and

$$E_P = \frac{1}{2}T(\frac{\partial y}{\partial x})^2 = \frac{1}{2}Tk^2 y_o^2 \sin^2(kx - \omega t) \quad (6)$$

These are equal, since

$$v^2 = \frac{\omega^2}{k^2} = \frac{T}{\rho} \tag{7}$$

The total energy density is therefore

$$E = E_k + E_P = \rho\omega^2 y_o^2 \sin^2(kx - \omega t) \tag{8}$$

Now

$$\rho\omega^2 y_o^2 = 0.1 \times (2\pi \times 3)^2 (0.5)^2 J/m = 8.88 J/m \tag{9}$$

while

$$k = \omega\sqrt{\frac{\rho}{T}} = 2\pi \times 3 \times \sqrt{\frac{0.1}{0.9}} m^{-1} = 2\pi m^{-1} \tag{10}$$

so

$$E = 8.88 \sin^2(2\pi x - 6\pi t) J/m \tag{11}$$

(c) (2) Calculate the average energy density in the string.

The average energy density is

$$< E >= 8.88 < \sin^2(2\pi x - 6\pi t) > J/m = 4.44 J/m$$

(d) (3) Calculate the average power propagating along the string.

The average power is

$$< P >=< E > v =< E > \sqrt{\frac{T}{\rho}} = 4.44 \times \sqrt{\frac{.9}{.1}} W = 13.32 W \tag{12}$$

(e) (3) Give an expression for the vertical component of the force the person exerts on the string.

The person must exert an equal and opposite force to that exerted by the string in the $y-$ direction.

$$f_y = -T\frac{\partial y}{\partial x} = T y_o k \sin(kx - \omega t) = 2.83 \sin(2\pi x - 6\pi t) N \tag{13}$$

(f) (3) Give an expression for the power produced by the person moving the end of the string.

The power produced is just the force times the velocity; that is,

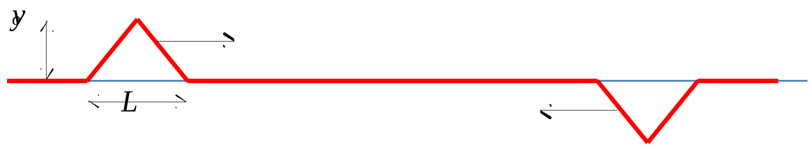
$$P = f_y \frac{\partial y}{\partial t} = f_y \omega y_o \sin(kx - \omega t) = 26.65 \sin^2(2\pi x - 6\pi t) W \tag{14}$$

(g) (2) Calculate the average power produced by the person moving the end of the string.

The average power is

$$< P >= 26.65 < \sin^2(3.37x - 18.9t) > W = 13.32 W \tag{15}$$

2. (15)The string below has mass density $\rho = 0.4 kg/m$ and tension $T = 0.9N$. It has two triangular waves propagating on it, as shown. $y_o = 10cm$, and $L = 20cm$.



(a) (8)Give an expression for the potential energy density and the kinetic energy density for each wave when the waves are far apart.

Let us denote the right triangular pulse by

$$y_l = f(x - vt) \tag{16}$$

and the right by

$$y_r = -f(-x - vt) \tag{17}$$

and we see at once that

$$\frac{\partial y_l}{\partial t} = -v\frac{\partial y_l}{\partial x} \tag{18}$$

and

$$\frac{\partial y_r}{\partial t} = v\frac{\partial y_r}{\partial x} \tag{19}$$

In general, we have that

$$y = y_l + y_r = f(x - vt) - f(-x - vt) \tag{20}$$

When the waves are far apart, the potential energy density of y_l is

$$\mathcal{E}_{pot} = \frac{1}{2}T(\frac{\partial y_l}{\partial x})^2 = \frac{1}{2}T(\frac{y_o}{L/2})^2 \tag{21}$$

$$= 0.45 J/m \tag{22}$$

and its kinetic energy density is

$$\mathcal{E}_{kin} = \frac{1}{2}\rho(\frac{\partial y_l}{\partial t})^2 = \frac{1}{2}\rho v^2(\frac{y_o}{L/2})^2 \tag{23}$$

$$= \frac{1}{2}T(\frac{y_o}{L/2})^2 = 0.45 J/m \tag{24}$$

Similarly, for y_r, we have the potential energy density

$$\mathcal{E}_{pot} = \frac{1}{2}T(\frac{\partial y_r}{\partial x})^2 = \frac{1}{2}T(\frac{y_o}{L/2})^2 \tag{25}$$

$$= 0.45 J/m \tag{26}$$

and its kinetic energy density is

$$\mathcal{E}_{kin} = \frac{1}{2}\rho(\frac{\partial y_r}{\partial t})^2 = \frac{1}{2}\rho v^2(\frac{y_o}{L/2})^2 \tag{27}$$

$$= \frac{1}{2}T(\frac{y_o}{L/2})^2 = 0.45J/m \tag{28}$$

The total energy in the string is the energy density times the length, and we have

$$\mathcal{E}_{TOT} = 2T(\frac{y_o}{L/2})^2 L = 0.36J \tag{29}$$

(b) (7) Give an expression for the potential energy density and the kinetic energy density in the string for when the waves are right on top of each other, and the displacement of the string is $y = 0$.

Again, in general we have that

$$y = y_l + y_r = f(x - vt) - f(-x - vt) \tag{30}$$

When the pulses are on top of each other, the potential energy density is

$$n\mathcal{E}_{pot} = \frac{1}{2}T(\frac{\partial y}{\partial x})^2 = 0 \tag{31}$$

The kinetic energy is

$$\mathcal{E}_{kin} = \frac{1}{2}\rho(\frac{\partial y_l}{\partial t} + \frac{\partial y_r}{\partial t})^2 = \frac{1}{2}\rho(-v\frac{\partial y_l}{\partial x} + v\frac{\partial y_r}{\partial x})^2 \tag{32}$$

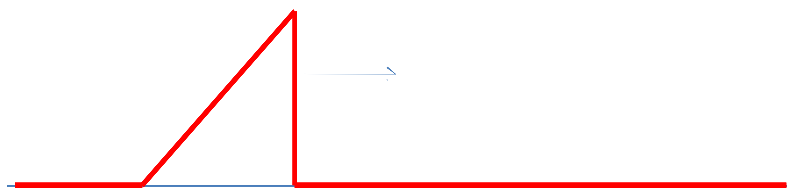
$$= \frac{1}{2}\rho(-v\frac{y_o}{L/2} - v\frac{y_o}{L/2})^2 \tag{33}$$

$$= 2\rho v^2(\frac{y_o}{L/2})^2 = 2T(\frac{y_o}{L/2})^2 = 1.80J/m \tag{34}$$

and the total energy is, as before,

$$\mathcal{E}_{TOT} = 2T(\frac{y_o}{L/2})^2 L = 0.36J \tag{35}$$

3. (11)The wave shown below is travelling on a string with mass density $\rho = 0.4kg/m$ and tension $T = 0.9N$. It is incident on a string with $\rho = 0.1kg/m$ and tension $T = 0.4N$.



Triangular wave on a string

(8) If the amplitude of the incident wave is $1.0cm$, what will be the amplitudes of the reflected and transmitted waves?

The impedances of the strings are

$$Z_1 = \sqrt{T_1\rho_1} = \sqrt{0.9 \times 0.4} = 0.6kg/s \tag{36}$$

and

$$Z_2 = \sqrt{T_2\rho_2 = \sqrt{0.4 \times 0.1} = 0.2kg/s} \tag{37}$$

The reflection coefficient is

$$r = \frac{Z_1 - Z_2}{Z_1 + Z_2} = \frac{1}{2} \tag{38}$$

and the transmission coefficient is

$$t = \frac{2Z_1}{Z_1 + Z_2} = \frac{3}{2} \tag{39}$$

The reflected wave amplitude is therefore

$$r \times 1.0cm = 0.5cm \tag{40}$$

and the transmitted wave amplitude is

$$t \times 1.0cm = 1.5cm \tag{41}$$

1. (a) (3)Sketch the transmitted and reflected waves.
   The velocities are

$$v_1 = \sqrt{\frac{T_1}{\rho_1}} = 1.5m/s \tag{42}$$

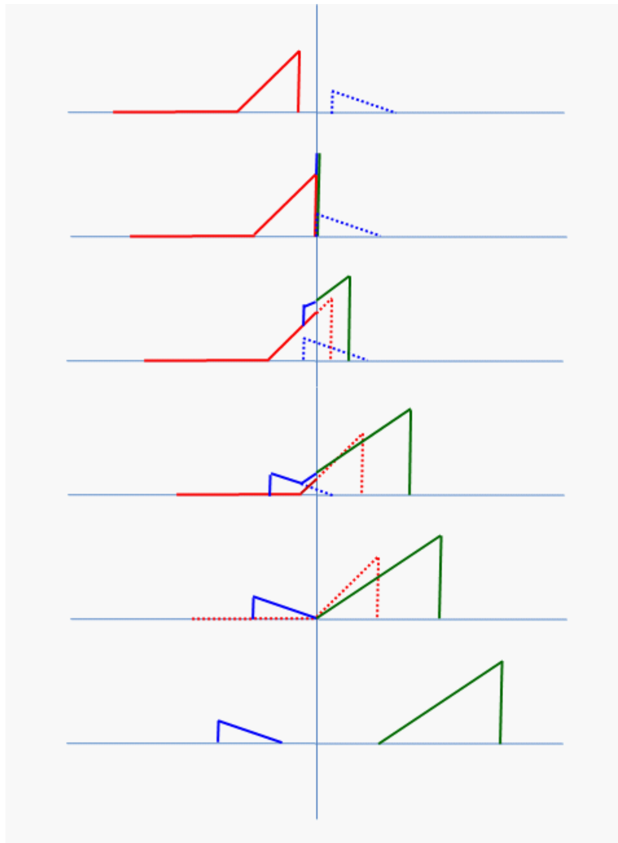


Figure 1: Details of waves on interface.

and

$$v_2 = \sqrt{\frac{T_2}{\rho_2}} = 2m/s \tag{43}$$

and so

$$v_2 = v_1\sqrt{\frac{T_2\rho_1}{T_1\rho_2}} = \frac{4}{3}v_1 \tag{44}$$

The transmitted and reflected waves are shown in detail below. (This is a bit more detail than what I expected from you.) It is useful to imagine a virtual reflected wave (dashed blue) and a virtual incident wave (dashed red). The solid lines show the string position. Below, we show only the wave amplitude at different times.

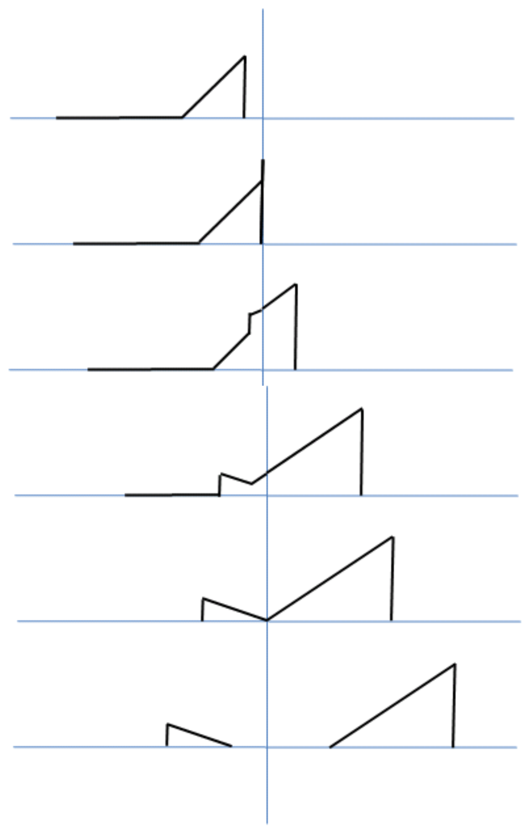


Figure 2: String amplitude at different times.