Optics I: Theory CPHY 6/72250
Assignment 2.

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Due: Oct. 4, 2016

1. (15)(To do this problem, you must have a clear idea what a principal point is; we did not cover this in class. So find out what it is on the web, and once you know what it is, go ahead with the problem. The figure is not very clear, I'm afraid.) Show that a SELFOC slab of length $d < \pi/2\alpha$ and refractive index $n = n_0(1 - \frac{1}{2}\alpha^2 y^2)$ in air acts as a cylindrical lens (a lens with focusing powerin the $x - z$ plane) of focal length

$$f = \frac{1}{n_0 \alpha \sin(\alpha d)} \tag{1}$$

We start with the ray equation in the paraxial approximation,

$$\frac{\partial^2 y}{\partial z^2} = \frac{1}{n}\frac{\partial n}{\partial y} \tag{2}$$

and

$$\frac{\partial^2 y}{\partial z^2} \simeq -\alpha^2 y \tag{3}$$

whose solution is

$$y = y_0 \cos \alpha z \tag{4}$$

The entrance surface is at $x = 0$, and at the exit surface, at $x = d$, we have

$$y = y_0 \cos \alpha d \tag{5}$$

The slope at $x = d$ is

$$y' = -y_0 \alpha \sin \alpha d \tag{6}$$

Now the slope y' is just $\tan \theta_1$ where θ_1 is the angle of incidence on the glass side. Then, Snell's law gives

$$n_0 \sin \theta_1 = n_2 \sin \theta_2 \tag{7}$$

If the angles are small, ($d < \pi/2\alpha$) we have

$$n_0\theta_1 = n_2\theta_2 \tag{8}$$

and the slope y' on the air side is just $\tan\theta_2 \simeq \theta_2$ and

$$\tan\theta_2 = \frac{n_0}{n_2}\theta_1 = -y_0\frac{n_0}{n_2}\alpha\sin\alpha d \tag{9}$$

The ratio of the height at the exit surface and the horizontal distance f is

$$\tan\theta_2 = \frac{y_0\cos\alpha d}{f} = y_0\frac{n_0}{n_2}\alpha\sin\alpha d \tag{10}$$

and

$$f = \frac{\cos\alpha d}{\frac{n_0}{n_2}\alpha\sin\alpha d} \tag{11}$$

and since $d < \pi/2\alpha$, cos$\alpha d \simeq 1$, and for air, $n_2 \simeq 1$, and we finally get

$$f = \frac{1}{n_0\alpha\sin\alpha d} \tag{12}$$

Show that the principal point H lies at a distance

$$AH = \frac{1}{n_0\alpha}\tan(\frac{\alpha d}{2}) \tag{13}$$

from the slab edge.

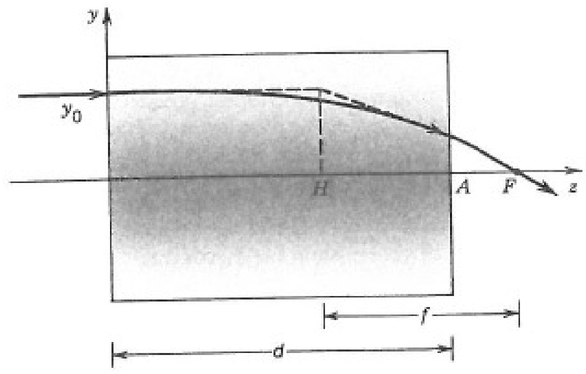
The prinicpal point is the intersection of the straight horizontal line (representing the ray from infinity) and the image ray, at the focal point, continued backwards as a straight line.

We know the slope of the line at the focal point, and its height at the exit surface. We can write at once that

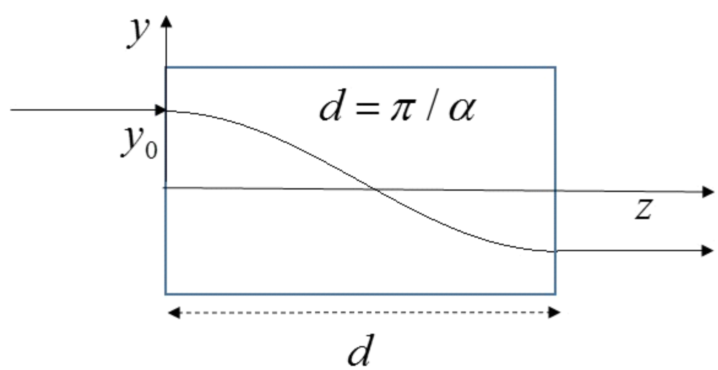
$$\tan\theta_2 = \frac{y_0 - y_0\cos\alpha d}{AH} = y_0\frac{n_0}{n_2}\alpha\sin\alpha d \tag{14}$$

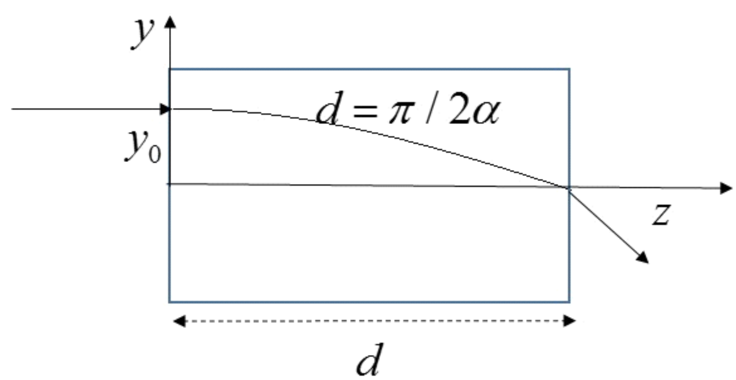
and

$$AH = \frac{1-\cos\alpha d}{n_0\alpha\sin\alpha d} = \frac{1}{n_0\alpha}\frac{1-(\cos^2(\alpha d/2)-\sin^2(\alpha d/2))}{2\sin(\alpha d/2)\cos(\alpha d/2)} = \frac{1}{n_0\alpha}\tan(\alpha d/2) \tag{15}$$

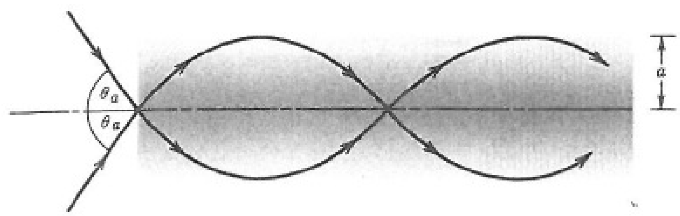


Sketch the ray trajectories in the special cases $d = \pi/\alpha$ and $\pi/2\alpha$.Trajectories are shown below.





2. (8)Consider a graded index fiber with refractive index $n = n_1(1 - \frac{1}{2}\alpha^2 y^2)$ and radius a. A ray is incident from material with index n_0 at its center, making an angle θ_0 with the fiber axis.



Show that, in the paraxial approximation, the numerical aperture is

$$NA = n_0 \sin\theta_0 \approx n_1 \alpha a \tag{16}$$

where θ_0 is the angle for which the ray trajectory is confined within the fiber.

We again start with the ray equation in the paraxial approximation

$$\frac{\partial^2 y}{\partial z^2} = \frac{1}{n}\frac{\partial n}{\partial y} \tag{17}$$

and

$$\frac{\partial^2 y}{\partial z^2} \simeq -\alpha^2 y \tag{18}$$

The solution is

$$y = a \sin \alpha z \tag{19}$$

and the slope at the entrance surface is

$$y' = a\alpha \simeq \theta_1 \tag{20}$$

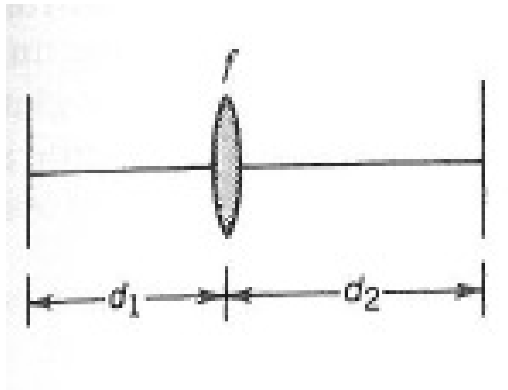
and using Snell's law, we have

$$n_0 \sin\theta_0 = n_1 \sin\theta_1 \tag{21}$$

and approximately

$$NA = n_0 \sin\theta_0 \simeq n_1 \alpha a \tag{22}$$

3. (12)Derive an expression for the transfer matrix of a system comprised of free space/thin lens/free space as shown below.



Show that if the imaging condition

$$\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f} \tag{23}$$

is satisfied, all rays originating from a single point y_1 in the input plane reach the output plane at the single point y_2, regardless of their angles. Also show that if $d_2 = f$, all parallel incident rays are focused by the lens onto a single point in the output plane.

We begin by writing down the propagation matrices for all the elements. First, we have propagation in free space through distance d_1 ,

$$M_1 = \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \tag{24}$$

then we have transmission through the lens

$$M_2 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \tag{25}$$

then we have propagation in free space through distance d_2

$$M_3 = \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \tag{26}$$

and we have that the total propagation matrix, connecting the output with the input is

$$M_T = M_3 M_2 M_1 \tag{27}$$

or

$$M_T = \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \tag{28}$$

and

$$M_T = \begin{bmatrix} 1 - \frac{1}{f}d_2 & d_2 - d_1\left(\frac{1}{f}d_2 - 1\right) \\ -\frac{1}{f} & 1 - \frac{1}{f}d_1 \end{bmatrix} \tag{29}$$

If the input ray vector is

$$\begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} \tag{30}$$

the output is

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{f}d_2 & d_2 - d_1\left(\frac{1}{f}d_2 - 1\right) \\ -\frac{1}{f} & 1 - \frac{1}{f}d_1 \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} \tag{31}$$

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \theta_1\left(d_2 - d_1\left(\frac{1}{f}d_2 - 1\right)\right) - y_1\left(\frac{1}{f}d_2 - 1\right) \\ -\frac{1}{f}y_1 - \theta_1\left(\frac{1}{f}d_1 - 1\right) \end{bmatrix} \tag{32}$$

: If the imaging condition

$$\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f} \tag{33}$$

is satisfied, then

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} -y_1\frac{d_2}{d_1} \\ -\frac{1}{f}y_1 - \theta_1\frac{d_1}{d_2} \end{bmatrix} \tag{34}$$

That is, all rays originating from (y_1, θ_1) go to through $y_2 = -y_1\frac{d_2}{d_1}$ regardless of direction θ_1.

: If $d_2 = f$, then $d_1 = \infty$, and

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{d_2}y_1 - \theta_1\frac{d_1}{d_2} \end{bmatrix} \tag{35}$$

If incident rays are parallel to the optic axis, then $\theta_1 = 0$, and

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{d_2}y_1 \end{bmatrix} \tag{36}$$

and all rays go through the point $y_2 = 0, z = d_2$ regardless of y_1.