Optics I: Theory CPHY 6/72250
Assignment 1 - Solutions.

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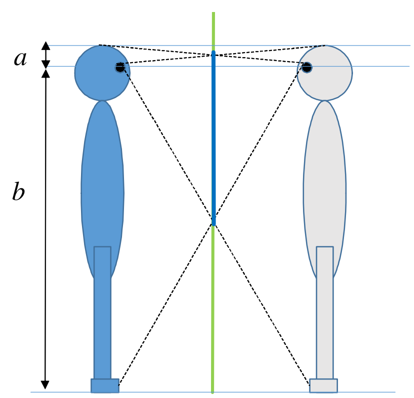
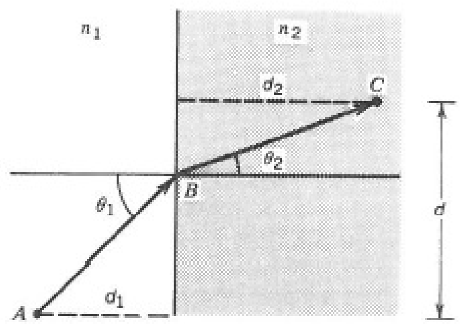
1. What length plane mirror must you purchase to see your full height when it is mounted in a vertical position? (Do not assume that you have eyes on the top of your head.) Make a sketch, and label distances.Consider

   the figure shown.



First, we assume that the front of the person is in a plane parallel to the mirror, that is, the points in the object are equidistant from the mirror. We rely on the law of reflection; the angle of incidence is equal to the angle of reflection. Therefore, to see the lowest point in the reflection, the mirror must extend the distance $b/2$ below the level of the eyes, as shown in the figure. Similarly, to see the highest point on the reflection, the mirror must extend the distance $a/2$ above your eyes. So the mirror must have length $(a+b)/2$. Your height is $a+b$. The length of the mirror must therefore be half your height. the result is independent of the object distance; that is, the distance from the mirror.

2. Consider the planar interface shown.



Show that the optical path length from A to C is minimized if

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{1}$$

This is Snell's Law.
The optical path length OPL travelled by the ray from A to B is just

$$OPL = \frac{n_1 d_1}{\cos\theta_1} + \frac{n_2 d_2}{\cos\theta_2} \tag{2}$$

and

$$d_1 \tan\theta_1 + d_2 \tan\theta_2 = d \tag{3}$$

and d is a constant. Brute force method: We can eliminate one of the angles, say θ_2.. We write

$$\tan\theta_2 = \frac{d}{d_2} - \frac{d_1}{d_2}\tan\theta_1 \tag{4}$$

or

$$1 + \tan^2\theta_2 = \frac{1}{\cos^2\theta_2} = 1 + (\frac{d}{d_2} - \frac{d_1}{d_2}\tan\theta_1)^2 \tag{5}$$

and so

$$OPL = \frac{n_1 d_1}{\cos\theta_1} + n_2 d_2 \sqrt{1 + (\frac{d}{d_2} - \frac{d_1}{d_2}\tan\theta_1)^2} \tag{6}$$

Now minimizing OPL with respect to θ_1 gives

$$0 = n_1 d_1 \frac{\sin\theta_1}{\cos^2\theta_1} + \frac{1}{2} n_2 d_2 \cos\theta_2 [-2(\frac{d}{d_2} - \frac{d_1}{d_2}\tan\theta_1)(\frac{d_1}{d_2})(\frac{1}{\cos^2\theta_1})] \tag{7}$$

Multiplying through by $\cos^2\theta_1$ and using Eq. 4,

$$0 = n_1 d_1 \sin\theta_1 - n_2 d_2 \cos\theta_2 \tan\theta_2 (\frac{d_1}{d_2}) \tag{8}$$

or

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{9}$$

(This is not an elegant method; it can be done more simply, as many of you have shown. I just did it this way just to show that the method always works.)

3. Consider a series of plane interfaces, all parallel. At the first, the index changes from n_0 to $n1$; at the second, it changes from n_1 to n_2; at the mth from n_{m-1} to n_m. If θ_m is the angle of refraction and θ_{m-1} is the angle of incidence at the mth interface, show that

$$n_0 \sin\theta_0 = n_m \sin\theta_m \tag{10}$$

This is really straightforward; we write Snell's Law in the form

$$\frac{n_0 \sin\theta_0}{n_1 \sin\theta_1} = 1 \tag{11}$$

$$\frac{n_1 \sin\theta_1}{n_2 \sin\theta_2} = 1 \tag{12}$$

$$\frac{n_2 \sin\theta_2}{n_3 \sin\theta_3} = 1 \tag{13}$$

$$\frac{n_{m-1} \sin\theta_{m-1}}{n_m \sin\theta_m} = 1 \tag{14}$$

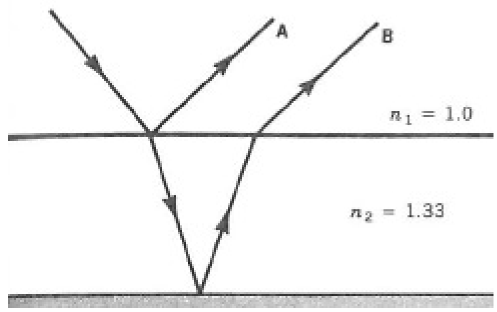
and we multiply all the equations together. Then we get, due to telescoping cancellation,

$$\frac{n_0 \sin\theta_0}{n_m \sin\theta_m} = 1 \tag{15}$$

or

$$n_0 \sin\theta_0 = n_m \sin\theta_m \tag{16}$$

4. A collimated laser beam shines on a tank of water. Part reflects from the surface (beam A) and part reflects from the bottom, and exits the water as beam B.



Show that the beams are parallel.

We note that the angle of transmission of the ray entering the water is θ_{t1} given by

$$n_0 \sin\theta_i = n_w \sin\theta_{t1} \tag{17}$$

the angle of incidence of the ray striking the water surface from below is also θ_{t1}. So the angle of transmission $\theta_{t2}$of the ray emerging from the water is

$$n_w \sin\theta_{t1} = n_0 \sin\theta_{t2} \tag{18}$$

Combining these, we get

$$\sin\theta_i = \sin\theta_{t2} \tag{19}$$

or

$$\theta_{t2} = \theta_i \tag{20}$$

and since the angle of reflection θ_r is the same as the angle of incidence (Law of Reflection), the two ray leaving the water surface must be parallel.