Optics I Theory CPHY 62250/72250
Assignment 3 Solutions

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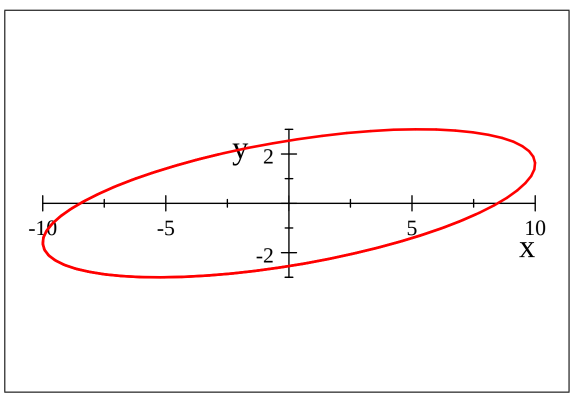
1. The electric field of a light wave is given by

$$\mathbf{E} = E_x \hat{\mathbf{x}} \cos(kz - \omega t + \phi_x) + E_y \hat{\mathbf{y}} \sin(kz - \omega t + \phi_y)$$

where $E_x = 10V/m$, $E_y = 3V/m$, $\phi_x = 33^o$ and $\phi_y = 65^o$.

(a) Plot the polarization ellipse.

This can be done with a simple parametric plot.



(b) What is the angle between the long axis of the ellipse and the $x-$axis? We have, from class notes, that if

$$\mathbf{E} = A\hat{\mathbf{x}} \cos(\omega t) + B\hat{\mathbf{y}} \cos(\omega t + \delta)$$

then

$$\tan 2\theta = \frac{2AB \cos \delta}{A^2 - B^2}$$

and

$$\theta = \frac{1}{2} \frac{2AB \cos \delta}{A^2 - B^2}$$

We have

$$\mathbf{E} = E_x \hat{\mathbf{x}} \cos(kz - \omega t + \phi_x) + E_y \hat{\mathbf{y}} \sin(kz - \omega t + \phi_y)$$

The polarization ellipse is independent of position, so we can write

$$\mathbf{E} = E_x \hat{\mathbf{x}} \cos(-\omega t + \phi_x) + E_y \hat{\mathbf{y}} \sin(-\omega t + \phi_y)$$

and it is independent of time, so we can write

$$\mathbf{E} = E_x \hat{\mathbf{x}} \cos(-\omega t) + E_y \hat{\mathbf{y}} \sin(-\omega t - \phi_x + \phi_y)$$

or

$$\mathbf{E} = E_x \hat{\mathbf{x}} \cos(-\omega t) + E_y \hat{\mathbf{y}} \cos(-\omega t - \phi_x + \phi_y - 90^\circ)$$

and

$$\mathbf{E} = E_x \hat{\mathbf{x}} \cos(\omega t) + E_y \hat{\mathbf{y}} \cos(\omega t + \phi_x - \phi_y + 90^\circ)$$

So we have $A = 10V/m$, $B = 3V/m$ and $\delta = \phi_x - \phi_y + 90^\circ$ or $\delta = 58^\circ$
We get at once that

$$\theta = 10.0^\circ$$

(c) What is the aspect ratio b/a of the ellipse?
We have from the class notes that

$$a^2 = \frac{1}{2} \frac{4A^2B^2 \sin^2\delta}{(A^2 + B^2) - \sqrt{(A^2 - B^2)^2 + 4A^2B^2\cos^2\delta}}$$

and

$$b^2 = \frac{1}{2} \frac{4A^2B^2 \sin^2\delta}{(A^2 + B^2) + \sqrt{(A^2 - B^2)^2 + 4A^2B^2\cos^2\delta}}$$

and we get at once

$$a = 10.1$$

and

$$b = 2.51$$

so the aspect ratio is

$$\frac{b}{a} = 0.249$$

2. π- polarized light is incident at $45\,^\circ$ on a water - glass interface. What fraction of the incident light energy is reflected?

   The reflection coefficient for π- polarized light is

$$r_\pi = \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_2 \cos\theta_1 + n_1 \cos\theta_2}$$

   The refractive index of water is $n_w = 1.33$, and the refractive index of glass is $n_g = 1.52$. Assuming that light is incident from the water side, we have $\theta_1 = 45\,^\circ$, and $\sin\theta_2 = (n_w/n_g)\sin\theta_1$ from Snell's law.
   We then have

$$r_\pi = \frac{n_g \cos\theta_1 - n_w \sqrt{1 - \frac{n_w^2}{n_g^2}\sin^2\theta_i}}{n_g \cos\theta_1 + n_w \sqrt{1 - \frac{n_w^2}{n_g^2}\sin^2\theta_1}} = 1.4125 \times 10^{-2}$$

   This tells us that

$$\frac{E_r}{E_i} = r_\pi$$

   We know that the light energy is proportional to the square of the electric field, hence

$$\frac{I_r}{I_i} = r_\pi^2 = 2.00 \times 10^{-4}$$

   and $0,02\%$ of the light energy is reflected.

3. Explain why there is no reflection of light when the light is incident on a plane interface at Brewster's angle.

   In medium 1, $r$eflected light is due to radiation of the oscillating dipoles in medium 2, in which transmitted light propagates. If the wave vector of the reflected light is perpendicular to the wave vector of the transmitted light, it must be parallel to the electric field (that is, the dipole moments) in medium 2. Oscillating dipoles do not radiate light in the direction of polarization - hence there is no reflection.

4. σ-polarized light is incident at $45\,^\circ$ on a glass - air interface from the glass side. The interface is in the $x-z$ plane.

   The critical angle for total internal reflection is $n_g = 1.52$

   $$\theta_c = \sin^{-1}(\frac{n_a}{n_g}) = 41.1\,^\circ$$

   Since the angle of incidence is greater than the critical angle, we have TIR.

   (a) Give an expression for the electric field on the glass side.

   On the glass side, we have the incident electric field, and the reflected electric field.
   Assuming that the interface is along the x axis, and the normal to the interface is along the y axis, the incident electric field can be written as

   $$\mathbf{E}_i = E_i\hat{\mathbf{z}}e^{i(k_g(\frac{1}{\sqrt{2}}\hat{\mathbf{x}}-\frac{1}{\sqrt{2}}\hat{\mathbf{y}})\cdot\mathbf{r}-\omega t)}$$

   where $k_g = 2\pi n_g/\lambda_o$.
   The reflection coefficient is $\theta_1 = 45\,^\circ$

   $$\begin{aligned} r_\sigma &= \frac{n_g\cos\theta_1 - n_a\sqrt{1-\frac{n_g^2}{n_a^2}\sin^2\theta_1}}{n_g\cos\theta_1 + n_a\sqrt{1-\frac{n_g^2}{n_a^2}\sin^2\theta_1}} = \frac{1.074\,8 - 0.393\,95i}{1.074\,8 + 0.393\,95i} \\ &= 0.763\,13 - 0.646\,25i \\ &= e^{-i40.3\,^\circ} \end{aligned}$$

   Note that the magnitude of r_σ is 1, but there is a phase, and multiplying the incident light amplitude by r_σ gives for the reflected light

   $$\mathbf{E}_r = E_i\hat{\mathbf{z}}e^{i(k_g(\frac{1}{\sqrt{2}}\hat{\mathbf{x}}-\frac{1}{\sqrt{2}}\hat{\mathbf{y}})\cdot\mathbf{r}-\omega t-40.3\,^\circ)}$$

   The electric field on the glass side is therefore the sum of the incident and reflected fields; that is

   $$\mathbf{E}_g = E_i\hat{\mathbf{z}}e^{i(k_g(\frac{1}{\sqrt{2}}\hat{\mathbf{x}}-\frac{1}{\sqrt{2}}\hat{\mathbf{y}})\cdot\mathbf{r}-\omega t)} + E_i\hat{\mathbf{z}}e^{i(k_g(\frac{1}{\sqrt{2}}\hat{\mathbf{x}}+\frac{1}{\sqrt{2}}\hat{\mathbf{y}})\cdot\mathbf{r}-\omega t-40.3\,^\circ)}$$

   Note that we can write this as

   $$\begin{aligned} \mathbf{E}_g &= E_i\hat{\mathbf{z}}e^{i(k_g(\frac{1}{\sqrt{2}}\hat{\mathbf{x}}-\frac{1}{\sqrt{2}}\hat{\mathbf{y}})\cdot\mathbf{r}-\omega t)}(1+e^{-i40.3\,^\circ}) \\ E_g &= E_i(1+0.763\,13-0.646\,25i) \\ &= E_i(1.763\,13-0.646\,25i) \end{aligned}$$

   (b) Give an expression for the electric field on the air side.

   The transmission coefficient is

   $$t_\sigma = \frac{2n_g\cos\theta_1}{n_g\cos\theta_1 + n_a\sqrt{1-\frac{n_g^2}{n_a^2}\sin^2\theta_1}} = 1.763\,1 - i0.646\,25i$$

which tells us that the magnitude of the field on the air side

$$E_a = E_i(1.763\,1 - 0.646\,25i)$$

This is what we expect, since the tangential component of the **E**-field is continuous, and we have σ polarization.

We can think of the wave vector of the transmitted light as

$$\mathbf{k}_t = \frac{2\pi n_a}{\lambda_0}(\sin\theta_2, \cos\theta_2)$$

which we can write as

$$\mathbf{k}_t = \frac{2\pi n_a}{\lambda_0}\sin\theta_2\hat{\mathbf{x}} - \frac{2\pi n_a}{\lambda_0}\cos\theta_2\hat{\mathbf{y}}$$

We know that from Snell's law, we have

$$\sin\theta_2 = \frac{n_g}{n_a}\sin\theta_1$$

and

$$\cos\theta_2 = \sqrt{1 - \frac{n_g^2}{n_a^2}\sin^2\theta_1}$$

and so

$$\mathbf{k}_t = \frac{2\pi n_a}{\lambda_0}\frac{n_g}{n_a}\sin\theta_1\hat{\mathbf{x}} - \frac{2\pi n_a}{\lambda_0}\sqrt{1 - \frac{n_g^2}{n_a^2}\sin^2\theta_1}\hat{\mathbf{y}}$$

or

$$\mathbf{k}_t = k_g\frac{1}{\sqrt{2}}\hat{\mathbf{x}} - 0.393\,95k_a i\hat{\mathbf{y}}$$

So the real part of the wave vector is along the x axis, and it has the same magnitude as the x component of the **E**-field on the glass side. This is the direction of the wave propagation on the air side. There is exponential decay as we move away from the interface in the $-y$ direction. The field on the air side is therefore

$$\mathbf{E}_a = E_i(1.763\,1 - 0.646\,25i)\hat{\mathbf{z}}e^{i((k_g\frac{1}{\sqrt{2}}\hat{\mathbf{x}} - i0.393\,95k_a\hat{\mathbf{y}})\cdot\mathbf{r} - \omega t)}$$

or

$$\mathbf{E}_a = E_i(1.763\,1 - 0.646\,25i)\hat{\mathbf{z}}e^{0.393\,95k_a y}e^{i((k_g\frac{1}{\sqrt{2}}x - \omega t)}$$

So the wave propagates in the x-direction along the interface, and decays exponentially with decay constant $1/0.393\,95k_a$. This is a typical evanescent wave. Note that we could have written the complex amplitude as a real amplitude and a phase.

(c) Give an expression for the magnetic field on the air side.

The magnetic field can be obtained from the Maxwell equation

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

which gives

$$\mathbf{H} = -\frac{1}{\mu_0} \int \nabla \times \mathbf{E} dt$$

Now

$$\nabla \times \mathbf{E} = i\mathbf{k} \times \mathbf{E}$$

and

$$\begin{aligned} i\mathbf{k} \times \mathbf{E} &= i \begin{matrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ k_{ax} & k_{ay} & 0 \\ 0 & 0 & E_a \end{matrix} \\ &= ik_{ay}E_a\hat{\mathbf{x}} - ik_{ax}E_a\hat{\mathbf{y}} \end{aligned}$$

Integration over time gives

$$\begin{aligned} \mathbf{H}_a &= -\frac{1}{\mu_0}(-\frac{1}{i\omega})i(k_{ay}\hat{\mathbf{x}} - k_{ax}\hat{\mathbf{y}})E_a \\ &= \frac{k_a}{\mu_0\omega}(-\cos\theta_2\hat{\mathbf{x}} - \sin\theta_2\hat{\mathbf{y}})E_a \\ &= \frac{E_a}{Z_a}(-\cos\theta_2\hat{\mathbf{x}} - \sin\theta_2\hat{\mathbf{y}}) \end{aligned} \tag{1}$$

Interestingly, here we have two components of $\mathbf{H}_a$, one in the $\hat{\mathbf{y}}$ direction, perpendicular to the $\mathbf{E}_a$ field and to the propagation direction $\hat{\mathbf{x}}$, and another along the propagation direction $\hat{\mathbf{x}}$. So we real have energy propagation along the interface, as described by $\mathbf{E}_a \times H_{ay}\hat{\mathbf{y}}$, (this energy current was set up when the light was first turned on; all the incident energy on the interface is reflected), but there is no net energy energy propagation in the $-\hat{\mathbf{y}}$ direction. Although $\mathbf{E}_a \times H_{ax}\hat{\mathbf{x}}$ is not zero, $H_{ax} = -\frac{E_a}{Z_a}\cos\theta_2$ is imaginary, which means it is out of phase with the electric field by 90°, and the average of $\cos\omega t \sin\omega t$ is zero. So energy flows back and forth along the y-axis on the air side, but there is no energy transport in that direction.