Optics I Theory CPHY 62250/72250
Assignment 2 Solutions

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1. Starting from Maxwell's equations, derive the wave equation for the magnetic field $\mathbf{H}$ if there are no free charges. Show all steps, and state all assumptions made.

We begin with

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

and assuming that ε is independent of time, we have

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

We take the curl again, and if the material is isotropic (ε is a scalar, not a tensor) we have

$$\nabla \times \nabla \times \mathbf{H} = \varepsilon \frac{\partial}{\partial t} \nabla \times \mathbf{E}$$

and it follows that

$$\nabla(\nabla \cdot \mathbf{H}) - \boldsymbol{\nabla}^2 \mathbf{H} = \varepsilon \frac{\partial}{\partial t} \nabla \times \mathbf{E}$$

and substiution gives

$$\nabla(\nabla \cdot \mathbf{H}) - \boldsymbol{\nabla}^2 \mathbf{H} = -\varepsilon \frac{\partial^2}{\partial t^2} \mathbf{B}$$

If μ is independent of time,

$$\nabla(\nabla \cdot \mathbf{H}) - \boldsymbol{\nabla}^2 \mathbf{H} = -\varepsilon \mu \frac{\partial^2}{\partial t^2} \mathbf{H}$$

and if the material is isotropic, μ is a scalar, and we have

$$\boldsymbol{\nabla}^2\mathbf{H} = \varepsilon\mu\frac{\partial^2}{\partial t^2}\mathbf{H}$$

which is the wave equation.

2. Show that $\mathbf{H} = \mathbf{H}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ satisfies the wave equation.

Substituting $\mathbf{H} = \mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{r}-\omega t}$ into the wave equation

$$\boldsymbol{\nabla}^2\mathbf{H} = \varepsilon\mu\frac{\partial^2}{\partial t^2}\mathbf{H}$$

gives at once

$$k^2\mathbf{H} = \varepsilon\mu\frac{\omega^2}{k^2}\mathbf{H}$$

so the solution satisfies the wave equation so long as

$$\frac{\omega}{k} = v = \frac{1}{\sqrt{\varepsilon\mu}}$$

(a) What is the relation between $\mathbf{H}_0$ and $\mathbf{k}$?

$\nabla\cdot\mathbf{B}$ implies that $\mathbf{k}\cdot\mathbf{H}_o = 0$. That is $\mathbf{H}_o$ is perpendicular to the direction of propagation,

(b) What is the relation betwen $\mathbf{H}_0$ and $\mathbf{E}_0$?

$\nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t}$ implies that $\mathbf{E}_0 = -Z\hat{\mathbf{k}}\times\mathbf{H}_0$ where $Z = \sqrt{\frac{\mu}{\varepsilon}}$ is the impedance.

3. Consider a laser pointer in air, emitting light at $\lambda = 635nm$ with $5mW$ of power into a beam with circular cross-section and $4mm$ diameter. Assume that the beam is uniform, and that $\varepsilon_r = \mu_r = 1$.

   (a) What is the magnitude of the electric field $\mathbf{E}?\omega t$

   The average power/area is the intensity, which is the time average of the magnitude of the Poynting vector

$$I =< S >=< E_0 \cos(\omega t) H_0 \cos(\omega t) >= \frac{1}{2} E_0 H_0 = \frac{1}{2} \frac{E_0^2}{Z_0}$$

   Solving for E_0, we get

$$E_0 = \sqrt{2Z_0 I}$$

   Now

$$I = \frac{W}{\pi r^2}$$

   so

$$E_0 = \sqrt{2Z_0 \frac{W}{\pi r^2}} = \sqrt{2 \times 377 \times \frac{5 \times 10^{-3}}{\pi \times (2 \times 10^{-3})^2}} = 548V/m$$

   :

   (b) What is the magnitude of the electric displacement $\mathbf{D}$?

   We have that

$$\mathbf{D} = \varepsilon \mathbf{E}$$

   so

$$D_0 = 8.85 \times 10^{-12} F/m \times 548V/m = 4.84 \times 10^{-9} C/m^2$$

   in the same direction as the electric field.

(c) What is the magnitude of the magnetic field **H**?

Since the magnetic field magnitude is $H = E/Z$, we have that

$$H_0 = \frac{E_0}{Z} = \frac{548V/m}{377\Omega} = 1.45A/m$$

(d) What is the magnitude of the magnetic flux density **B**?

Since $B = \mu H$ we have that

$$\begin{aligned} B &= 4\pi \times 10^{-7} H/m \times 1.45A/m \\ &= 1.8221 \times 10^{-6} T \end{aligned}$$

(e) How many photons are emitted by the pointer every second?

We have $P = 5mW$ of power; which is energy per second. Since the energy of each photon is $h\nu$, the number of photons emitted per second is

$$\begin{aligned} n &= \frac{P}{h\nu} = \frac{P\lambda}{hc} = \frac{5 \times 10^{-3} \times 635 \times 10^{-9}}{6.63 \times 10^{-34} \times 3 \times 10^{8}} s^{-1} \\ &= 1.60 \times 10^{16} s^{-1} \end{aligned}$$

(f) What is the force on the hand holding the laser pointer due to radiation pressure?

The force is the rate of change of momentum. Each photon carries away momentum h/λ, so the force is just

$$\begin{aligned} F &= n\frac{h}{\lambda} = \frac{P}{h\nu}\frac{h}{\lambda} = \frac{P}{c} \\ &= \frac{5\times 10^{-3}W}{3\times 10^{8}m/s} = 1.67\times 10^{-11}N \end{aligned}$$

or $16.7pN$.

(g) If the energy stored in the two AA batteries in the laser pointer is $25kJ$, how long can the laser stay on?If the power consumption of

the laser is $W = 5mW$, then the time t before the batteries run out is

$$t = \frac{25\times 10^{3}}{5\times 10^{-3}} = 5\times 10^{6}s$$

or about 58 days.

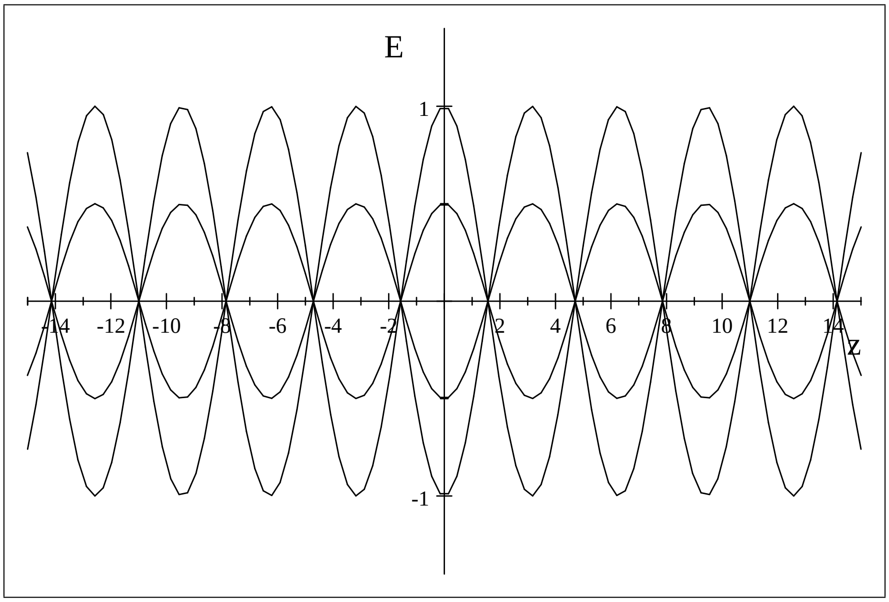
4. Consider two identical sinusoidal plane electromagnetic waves propagating in opposite directions.

(a) Sketch the amplitude of the electric field $\mathbf{E}$ in space at different times.

The total electric field amplitude si

$$\begin{aligned} E &= E_0\cos(kz-\omega t) + E_0\cos(kz+\omega t) \\ &= 2E_0\cos(kz)\cos(\omega t) \end{aligned}$$

and

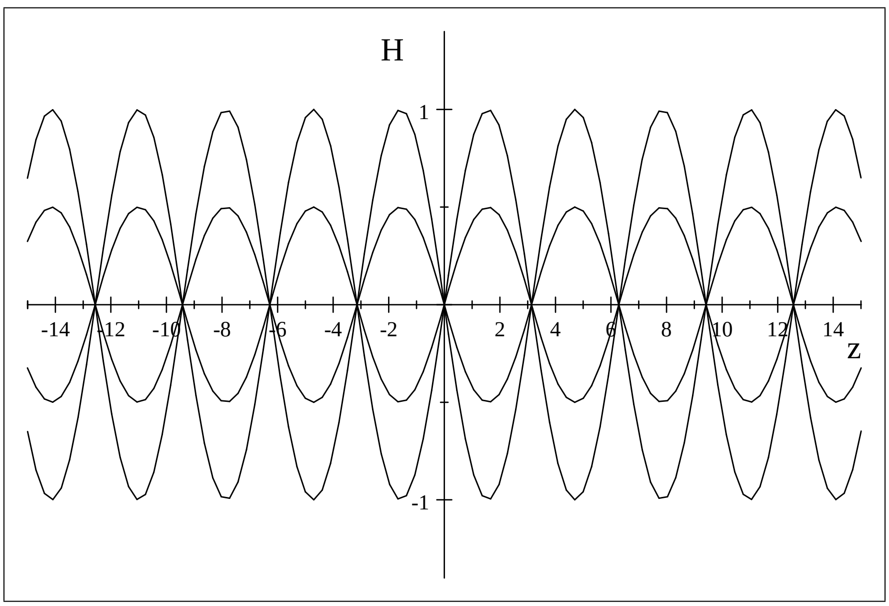


(b) Sketch the amplitude of the electric field **H** in space at different times.

The **H**-field of the counterpropagating wave will be in the opposite direction from that of the propagating one, so

$$
\begin{aligned}
H &= H_0 \cos(kz - \omega t) - H_0 \cos(kz + \omega t) \\
H &= 2H_0 \sin(kz) \sin(\omega t)
\end{aligned}
$$

and



(c) Give and expression for the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$.

The Poynting vector is given by

$$\begin{aligned}
\mathbf{S} &= (\mathbf{E}_+ + \mathbf{E}_-) \times (\mathbf{H}_+ + \mathbf{H}_-) \\
&= \hat{\mathbf{x}} E_0 (\cos(kz)\cos(\omega t)) \times \hat{\mathbf{y}} H_0 (\sin(kz)\sin(\omega t)) \\
&= \hat{\mathbf{z}} E_0 H_0 \cos(kz)\sin(kz)\cos(\omega t)\sin(\omega t) \\
&= \hat{\mathbf{z}} \frac{1}{4} E_0 H_0 \sin(2kz)\sin(2\omega t)
\end{aligned}$$

The polyting vector therefore keeps changing sign with both time and position; its average is zero.

(d) Sketch the energy density, averaged over time, as function of the position.

The electric field energy density is given by

$$\begin{aligned}
\mathcal{E}_E/vol &= \frac{1}{2}\varepsilon E^2 \\
&= 2\varepsilon E_0^2 \cos^2(kz)\cos^2(\omega t)
\end{aligned}$$

The magnetic field energy density is given by

$$\begin{aligned}
\mathcal{E}_H/vol &= \frac{1}{2}\mu H^2 \\
&= \frac{1}{2}\mu H_0^2 \sin^2(kz)\sin^2(\omega t) \\
&= \frac{1}{2}\mu \frac{E_0^2}{Z^2} \sin^2(kz)\sin^2(\omega t) \\
&= \frac{1}{2}\varepsilon E^2 \sin^2(kz)\sin^2(\omega t)
\end{aligned}$$

The total energy density is

$$\mathcal{E}_T/vol = \mathcal{E}_E/vol + \mathcal{E}_H/vol = 2\varepsilon E_0^2[\cos^2(kz)\cos^2(\omega t) + \sin^2(kz)\sin^2(\omega t)]$$

and the time averaged energy density is

$$\begin{aligned}
\mathcal{E}_T/vol &= \varepsilon E_0^2[\cos^2(kz) + \sin^2(kz)] \\
&= \varepsilon E_0^2
\end{aligned}$$

independent of position